

Decomposition of Banach Spaces

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The purpose of this lecture is to discuss some open problems concerning the possibility of decomposing infinite-dimensional Banach spaces into direct sums. Typically for Banach space theory this discussion is presented on two levels: the level of general Banach spaces and the more concrete level of the classical Banach spaces. We state three central questions on decomposition and then give a brief survey of the present knowledge in these directions.

Let X be an infinite-dimensional Banach space.

(i) X is called *indecomposable* if there is no bounded linear projection P in X such that $\dim PX = \dim (I - P)X = \infty$.

(ii) X is called *prime* if for every bounded linear projection P on X with $\dim PX = \infty$ we have $PX \approx X$ (\approx denotes isomorphism *i.e.* linear homeomorphism).

(iii) X is called *primary* if for every bounded linear projection P on X at least one of the relations $PX \approx X$ or $(I - P)X \approx X$ holds.

Clearly, \Rightarrow prime \Rightarrow primary. The primary spaces are those spaces which cannot be represented as direct sums of "simpler" spaces.

Question 1. *Do there exist indecomposable Banach spaces?*

Question 2. *Are c_0 and l_p , $1 \leq p \leq \infty$, the only prime spaces?*

Question 3. *Which are the primary spaces? In particular, are the "classical" separable spaces primary?*

It is widely conjectured that the answer to Question 1 is negative. Unfortunately, however, there is no method known of constructing a non-trivial projection in a general separable infinite-dimensional Banach space. Only if one assumes quite strong special assumptions (*e.g.* that the space can be ordered as a boundedly complete linear lattice) is it possible (and then generally easy) to prove the existence of a non-trivial projection. Actually one does not know whether there is a Banach space X such that every bounded linear operator $T : X \rightarrow X$ is of the form $T = \lambda I + C$ with C compact and λ a scalar. My feeling is that if one could construct for every X a non trivial operator $T : X \rightarrow X$ then the same method should enable the construction of a non-trivial projection.

While practically nothing of significance is known concerning Question 1 in