

Some Remarks on Operator-Convex Functions

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Let \mathcal{H} be a Hilbert space; we denote convergence in the weak and strong operator topologies of $\mathfrak{B}(\mathcal{H})$ by \xrightarrow{w} and \xrightarrow{s} . A Borel function $\phi(\lambda)$ defined on the open interval $a < \lambda < b$ is said to be *operator-convex* on (a, b) if $\phi(A + B/2) \leq \frac{1}{2}[\phi(A) + \phi(B)]$ for all self-adjoint A, B with spectra in (a, b) . Such ϕ have been characterized in a number of ways, but for our purposes the most useful is that of J. Bendat and S. Sherman [1]: ϕ is operator-convex on (a, b) if and only if

$$(*) \quad \phi(\lambda) = r + s\lambda + \int \frac{\lambda^2 t}{t - \lambda} \mu(dt),$$

with μ a finite non-negative measure supported on the complement of (a, b) in $\mathbb{R}^1 \cup \{\infty\}$.

We wish to present here two theorems concerning operator-convex functions:

Theorem 1. *Let ϕ be an operator-convex function on (a, b) ; let A_α, A be self-adjoint with spectra in (a, b) . Suppose that $A_\alpha \xrightarrow{w} A$ and $\phi(A_\alpha) \xrightarrow{w} B$. Then $\phi(A) \leq B$.*

Theorem 2. *In Theorem 1, suppose further that $\phi(A) = B$. Then either $\phi(\lambda) = r + s\lambda$ or A_α in fact converges strongly to A .*

The considerations involved in these theorems arose in [2] (the fact that λ^{-1} is operator-convex on $(0, \infty)$ was used to show that the limit in the weak operator topology of a minimizing sequence of operators is itself a minimizing operator); in some cases, Theorem 2 shows that a weakly convergent minimizing sequence in fact converges strongly. We further hope that these theorems, after being suitably adapted for unbounded operators, may also prove useful in demonstrating some sort of strong convergence as a consequence of weak convergence of their resolvents to the correct limits.

Preliminary to giving the proofs of these theorems, we notice that if ϕ is unbounded at a or b , then by the Uniform Boundedness Theorem the spectra

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