

Transitive Algebras

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The Burnside Theorem [5, p. 276] states that the only transitive algebra of operators on a finite dimensional Hilbert space is the algebra of all operators. If the same is true of infinite dimensional Hilbert spaces, then the invariant subspace conjecture is also true. Since this conjecture is generally thought false, it follows that the generalized Burnside Theorem is probably not true. However, a counter-example to this stronger statement has not yet been found. The object of this note is to give a survey of some of the additional conditions on a transitive algebra that have been found to imply that the algebra contains all operators.

The results to be presented here are of an algebraic topological nature and find their purely algebraic antecedents in a density theorem of Jacobson [5]. An intermediate type of result was proved by Rickart [9] and Yood [13]. They showed that an algebraically transitive Banach algebra of operators on a complex vector space is strictly dense in the set of all linear transformation on the space.

The term *algebra* will be used to mean a weakly closed subalgebra containing the identity of the set $\mathfrak{B}(\mathcal{H})$ of all operators on a Hilbert space \mathcal{H} . The lattice $\text{Lat } \mathfrak{C}$ of a set of operators \mathfrak{C} is the collection of all (closed) subspaces invariant under all the operators in \mathfrak{C} . An algebra \mathfrak{A} is *transitive* if $\text{Lat } \mathfrak{A}$ is trivial, i.e. contains only $\{0\}$ and \mathcal{H} . The direct sum of n copies of a Hilbert space \mathcal{H} with itself will be denoted by $\mathcal{H}^{(n)}$, and the direct sum of n copies of an operator A on \mathcal{H} by $A^{(n)}$. The set of all $A^{(n)}$ with A in \mathfrak{A} will be denoted $\mathfrak{A}^{(n)}$. The connection between topology and lattices is provided by the following lemma [11], [8].

Lemma 1. *If \mathfrak{A} is an algebra and B is an operator such that $\text{Lat } \mathfrak{A}^{(n)} \subset \text{Lat } B^{(n)}$ for each n , then B is in \mathfrak{A} .*

A general density theorem, which appears implicitly in Theorem 1 of [8], follows from this lemma. A vector $\langle f_1, f_2, \dots, f_n \rangle$ in $\mathcal{H}^{(n)}$ will be called linearly independent if the set $\{f_1, f_2, \dots, f_n\}$ of its components is linearly independent.

Theorem 2. *If a transitive algebra \mathfrak{A} is such that for each $n > 1$ every cyclic subspace of $\mathfrak{A}^{(n)}$ generated by a linearly independent vector in $\mathcal{H}^{(n)}$ contains a non zero vector with a vanishing component, then $\mathfrak{A} = \mathfrak{B}(\mathcal{H})$.*

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