

# On the Stone–Weierstrass Theorem of $C^*$ -algebras

SHÔICHIRO SAKAI

**1. Introduction.** Let  $A$  be the  $C^*$ -algebra of all complex valued continuous functions vanishing at infinity on a locally compact space. The Stone–Weierstrass Theorem gives the conditions under which a  $C^*$ -subalgebra  $B$  coincides with  $A$ . A plausible non-commutative extension of the Stone–Weierstrass Theorem is

**Conjecture.** Let  $\mathfrak{A}$  be a  $C^*$ -algebra and let  $\mathfrak{B}$  be a  $C^*$ -subalgebra of  $\mathfrak{A}$ . Let  $P(\mathfrak{A})$  be the set of all pure states of  $\mathfrak{A}$  and let  $0$  be the identically zero function on  $\mathfrak{A}$ . Suppose that  $\mathfrak{B}$  separates  $P(\mathfrak{A}) \cup (0)$ , then  $\mathfrak{A} = \mathfrak{B}$ .

Kaplansky [9] proved a theorem equivalent to the conjecture for GCR  $C^*$ -algebras (equivalently, type I  $C^*$ -algebras [6], [13]). Glimm [5], Ringrose [10] and Akemann [1] gave some considerations related to this conjecture.

The purpose of this paper is to present another consideration to the conjecture. Unfortunately, we can not solve the problem completely; but the author feels that the results obtained here indicate strongly that the conjecture will be true for all separable  $C^*$ -algebras. Throughout the present paper, we shall deal with separable  $C^*$ -algebras only. The main tool to attack the problem is the reduction theory. As corollaries of our results, we shall show: (1) Let  $\mathfrak{A}$  be a separable  $C^*$ -algebra and let  $\mathfrak{B}$  be a uniformly hyperfinite  $C^*$ -subalgebra of  $\mathfrak{A}$ . Suppose that  $\mathfrak{B}$  separates  $P(\mathfrak{A}) \cup (0)$ , then  $\mathfrak{A} = \mathfrak{B}$ ; (2) A new proof of Kaplansky's Theorem in the separable case; (3) Let  $\mathfrak{A}$  be a separable  $C^*$ -algebra and let  $\mathfrak{B}$  be a  $C^*$ -subalgebra of  $\mathfrak{A}$ . Suppose that there exists a  $*$ -representation  $\{\pi, \mathfrak{S}\}$  of or such that  $\overline{\pi(\mathfrak{B})} \subset \overline{\pi(\mathfrak{A})}$  and the commutant of  $\pi(\mathfrak{B})$  is hyperfinite, where  $\overline{\pi(\cdot)}$  is the weak closure of  $\pi(\cdot)$ . Then,  $\mathfrak{B}$  can not separate  $P(\mathfrak{A}) \cup (0)$ ; (4) Let  $\mathfrak{A}$  be a separable  $C^*$ -algebra and let  $\mathfrak{B}$  be a  $C^*$ -subalgebra of  $\mathfrak{A}$ . Suppose that there exists a  $*$ -representation  $\{\pi, \mathfrak{S}\}$  of  $\mathfrak{A}$  such that  $\overline{\pi(\mathfrak{A})}$  is finite  $W^*$ -algebra and  $\overline{\pi(\mathfrak{B})} \subset \overline{\pi(\mathfrak{A})}$ , where  $\overline{\pi(\cdot)}$  is the weak closure of  $\pi(\cdot)$ . Then,  $\mathfrak{B}$  can not separate  $P(\mathfrak{A}) \cup (0)$ . As a corollary of this result, if  $G$  is a countable discrete group and if  $H$  is a proper subgroup of  $G$ , then there exist two elementary positive definite functions  $\varphi_1, \varphi_2$  on  $G$  such that  $\varphi_1 = \varphi_2$  on  $H$ , but  $\varphi_1 \neq \varphi_2$ .

University of Pennsylvania  
Date communicated: OCTOBER 15, 1970