

Operator Theory and Periodicity

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There are many formulations and proofs of the Bott Periodicity Theorem [1]. One formulation gives the homotopy groups of a certain group of operators on a Hilbert space: Let H be a separable Hilbert space over the complex numbers, let \mathfrak{B} denote the space of all bounded operators on H , let \mathfrak{J} be the ideal of compact operators and let \mathfrak{K} denote the set of all invertible operators in $I + \mathfrak{J}$. The (complex) periodicity theorem states that

$$\pi_j(\mathfrak{K}) = \begin{cases} 0 & j \text{ even} \\ \mathbb{Z} & j \text{ odd} \end{cases}.$$

(All sets of operators are in the norm, i.e. uniform, topology.) This formulation is easily seen to be equivalent to the original one of Bott [4] which states that

$$\pi_j(Gl(\infty, \mathbb{C})) = \begin{cases} 0 & j \text{ even} \\ \mathbb{Z} & j \text{ odd} \end{cases} \quad \text{where} \quad Gl(\infty, \mathbb{C}) = \lim_{n \rightarrow \infty} Gl(n, \mathbb{C})$$

and $Gl(n, \mathbb{C}) \rightarrow Gl(n+1, \mathbb{C})$ is given by

$$A \rightarrow \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right).$$

It is not hard to show that $Gl(\infty, \mathbb{C}) \rightarrow \mathfrak{K}$ by

$$A \rightarrow \left(\begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array} \right)$$

is a homotopy equivalence.

In this talk I should like to sketch still another formulation and proof of the Periodicity Theorem (due to Atiyah and Singer [2]) which is a mixture of operator theory and homotopy theory and which leads to an open problem I will pose later.

Let \mathfrak{F} denote the space of Fredholm operators on H and $\mathfrak{F}_1 = [A \in \mathfrak{F}; A^* = -A]$ and the essential spectrum of iA contains both positive and negative real num-