

Operator Theory of Degenerate Elliptic-Parabolic Equations

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This note is a brief exposition of a family of ideas that enables one to prove—as a case of a more general theorem—that certain second-order elliptic-parabolic differential operators on compact manifolds are Fredholm operators of index zero, even though they are sufficiently degenerate or otherwise ill-behaved that the standard methods for proving that fact are difficult or impossible to apply. A large and interesting class of such operators, which class constitutes our motivating example in the present paper, is that considered initially by L. Hörmander in [4]. Working locally (in an open set in \mathbb{R}^n), Hörmander showed that [4, Thm. 1.1, p. 149 ff.] if the operator can be written in the forms

$$\begin{aligned} Lu &= - \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} + cu \\ &= - \sum_{i=1}^r X_i^2 u + Yu + cu \end{aligned}$$

where in the first representation all coefficients are C^∞ functions and the matrix (a_{ij}) is everywhere positive semi-definite, while in the second representation all the X_i and Y are homogeneous first-order linear differential operators with C^∞ coefficients (C^∞ vector fields), and if the Lie algebra $\mathfrak{L}(X_1, \dots, X_r)$ or $\mathfrak{L}(X_1, \dots, X_r, Y)$ has rank n (the dimension of the underlying Euclidean space), then the operator is hypoelliptic in the sense that a solution of $Lu = \varphi$ in the sense of distributions is C^∞ wherever the distribution φ is. (In particular, null-solutions are C^∞ .) J. -M. Bony [1] then proved that Hörmander's hypotheses together with the assumption that the matrix (a_{ij}) does not vanish at any point imply that every point has a basis of neighborhoods for which the Dirichlet boundary value problem is solvable uniquely, with nonnegative solutions for nonnegative boundary values; moreover, a minimum principle and an appropriate version of Harnack's principle hold for C^∞ supersolutions and null-solutions of the operator L . Another interesting class of operators, whose

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