

Products of Two Involutions in the General Linear Group

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1. Introduction. In a recent paper, [2], Maria J. Wonenburger gives a necessary and sufficient condition for a linear transformation of a finite dimensional vector space over a field of characteristic unequal to 2 to be a product of two involutions. In this paper we obtain the same result without any restriction on the characteristic. The result is that an invertible linear transformation, T , of a finite dimensional vector space over a field is the product of two involutions if and only if T is similar to T^{-1} .

In this paper we assume the reader has some familiarity with elementary decomposition theory and similarity theory for matrices as found, for example in [1]. We denote the companion matrix for the (monic) polynomial $p(x)$ by C_p . The proof proceeds by decomposing (up to similarity) the matrix of a linear transformation which is similar to its inverse into the direct sum of matrices of two types, and showing that matrices of each type are products of involutions.

2. Decomposition; the non-symmetric factors.

Definitions. Let $g(x)$ be a monic polynomial of degree m over a field F , with constant term $a \neq 0$. Then let $\tilde{g}(x) = (x^m/a)g(1/x)$, so that $\tilde{g}(x)$ is monic, and $\tilde{g}(1/\omega) = 0$ if and only if $g(\omega) = 0$. If $g(x)$ is irreducible over F , so is $\tilde{g}(x)$; hence, either $\tilde{g}(x) = g(x)$ or else $g(x)$ and $\tilde{g}(x)$ have no root in common. We say $g(x)$ is *symmetric* if $g(x) = \tilde{g}(x)$.

Lemma 1. Let A be an invertible $n \times n$ matrix over a field F . If A is similar to A^{-1} , then A is similar to a direct sum, $B_1 \oplus \cdots \oplus B_r \oplus D_1 \oplus \cdots \oplus D_s$, where

$$B_i = C_{f_i}, \quad D_i = \left[\begin{array}{c|c} C_{g_i} & 0 \\ \hline 0 & C_{g_i}^{-1} \end{array} \right];$$

with each

$$f_i(x) = (p_i(x))^{n_i}$$