

Weak H -Rings

ROBERT W. WEST

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§1. Introduction. When examining ring-valued contravariant functors defined on a category of topological spaces one finds that the classifying space satisfies the ring axioms up to weak homotopy. So it is natural to ask if such strictly weak H -rings exist in the sense that the axioms are not satisfied up to ordinary homotopy. This paper, a sequel to [1], is devoted to constructing examples of such things.

To make precise our notions let space X be an H -abelian group under addition map $a: X \times X \rightarrow X$: this means that the axioms of an abelian group hold up to homotopy. An H -ring structure on X is a map $m: X \wedge X \rightarrow X$ which is homotopy distributive in the sense that the diagram

$$(1.1) \quad \begin{array}{ccc} X \wedge (X \times X) & \xrightarrow{\gamma} & (X \wedge X) \times (X \wedge X) \xrightarrow{m \times m} X \times X \\ \downarrow 1 \wedge a & & \downarrow a \\ X \wedge X & \xrightarrow{m} & X \end{array}$$

and its $a \wedge 1$ analogue are homotopy commutative. Here, $\gamma(x \wedge (y, z)) = (x \wedge y, x \wedge z)$. We then refer to X (together with m and a) as an H -ring. Of course, any null-homotopic map m is an H -ring structure so we shall only be interested in *non-trivial* H -ring structures, those that are not null-homotopic.

Similarly, m is a *weak H -ring structure* if (1.1) is weakly homotopy commutative. Since “homotopic” implies “weakly homotopic” we refer to m as a *strictly weak H -ring structure* if (1.1) fails to be homotopy commutative.

Notice that the definitions also make sense if X is a strictly weak H -abelian group under $a: X \times X \rightarrow X$; such X exist by (1.1) of [1]. (Since we automatically have homotopy inversion by page 430 of [1] we are assuming that the remaining group axioms hold up to strictly weak homotopy.)

We therefore have four types of structures to consider and all exist:

(1.2) Theorem. (i) *There exist path-connected H -abelian groups W and X with W admitting a (non-trivial) H -ring structure and X a strictly weak H -ring structure.*

(ii) *There exist path-connected strictly weak H -abelian groups Y and Z admitting respectively an H -ring and a strictly weak H -ring structure.*