

A Sharp Form of an Inequality by N. Trudinger

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§1. Statements.

1.) Let D be an open domain in the n -dimensional space R^n and $x \in R^n$, $n \geq 2$. $\dot{W}_q^1(D)$ denotes the Banach space obtained from the C^1 -functions $u(x)$ with compact support in D by completion with the norm

$$||u||_q = \left(\int_D |u_x|^q dx \right)^{1/q}.$$

Here u_x is the gradient of u and $|u_x|$ its norm in the Euclidean metric. It is well known that for $1 < q < n$ the elements of $\dot{W}_q^1(D)$ can be identified with elements in L_p ($1/p = 1/q - 1/n$) and this imbedding is continuous. For $q > n$ these elements can be identified with Hölder continuous function with Hölder exponent $1 - n/q$. This leaves the case $q = n$ and it is easy to find examples of unbounded functions in $\dot{W}_q^1(D)$. However, Trudinger was able to bound the integral

$$\int_D e^{\alpha u^p} dx, \quad p = \frac{q}{q-1},$$

for some positive α by a constant independent of u for all u in the unit ball $||u||_q \leq 1$ of $\dot{W}_q^1(D)$.

Trudinger's proof [9] of his interesting estimate made use of the power series expansion of the exponential function and some Sobolev estimates for the individual terms carefully observing the dependence on the exponent of the expansion. The aim of this note is to replace this proof by a more direct one and at the same time to obtain the best exponent α . It turns out there is a positive number α_n such that the above statement holds for $\alpha \leq \alpha_n$ and is false for $\alpha > \alpha_n$. The remarkable phenomenon is that the inequality still holds for the critical value itself.

Theorem 1. Let $u \in \dot{W}_n^1(D)$, $n \geq 2$ and

$$(1) \quad \int_D |u_x|^n dx \leq 1.$$

Then there exists a constant c which depends on n only such that

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