

The Scalar Curvature of Left-Invariant Riemannian Metrics

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Introduction. Suppose M is a manifold with a Riemannian metric g . If (M, g) is a Riemannian homogeneous space, then the scalar curvature R_g is constant on M , since R_g is a function on M invariant under isometries. Thus it is possible to define a real-valued function from the set of homogeneous Riemannian metrics on M by the correspondence $g \rightarrow R_g$. We call this the scalar curvature function for M .

In this article we consider the case when M is a Lie group G and we consider the scalar curvature function for G restricted to the set of left-invariant Riemannian metrics having a given fixed volume element on G . This set of metrics is in one to one correspondence with the space $SL(n, \mathbf{R})/SO(n)$, where $n = \dim G$ (cf. §2), and under this correspondence the scalar curvature function becomes a real analytic function on $SL(n, \mathbf{R})/SO(n)$.

In §1 some of the formulas for the Riemannian geometry of left-invariant metrics are derived. In §2 the scalar curvature function is defined and some of its elementary properties are explained.

In §3 a result of Nagano's [8] characterizing Einstein metrics on a compact manifold is generalized to a theorem stating that for a unimodular group G the left-invariant Einstein metrics on G exactly correspond to the critical points of the scalar curvature function for G (Theorem 1).

In §4 we derive some of the properties of the scalar curvature function for semi-simple Lie algebras. (Since locally isomorphic groups have the same scalar curvature function R , we often associate R with the Lie algebra rather than with the group). By restricting R to geodesics through the origin of $SL(n, \mathbf{R})/SO(n)$, one obtains functions of one real variable for which the critical points can sometimes be found and whose behavior at infinity can be determined. It is shown that R has at least two critical points, which correspond to non-isometric Riemannian structures, for every compact simple Lie algebra except $so(3)$ and

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