

The Theory of Semi-Analytic Vectors: A New Proof of a Theorem of Masson and McClary

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1. Introduction. Suppose A is a Hermitian operator on a Hilbert space, \mathfrak{H} . If $\psi \in D(A^n)$ for all n , we say $\psi \in C^\infty(A)$. Let us define:

Definition. A vector $\psi \in C^\infty(A)$ is called a *semi-analytic vector* for A if and only if

$$\sum_{n=0}^{\infty} \frac{\|A^n \psi\|}{(2n)!} t^n < \infty$$

for some $t > 0$.

The name comes from the similarity to the notion of analytic vector [7] for which the *stronger* condition that $\sum_{n=0}^{\infty} (\|A^n \psi\|/n!)t^n < \infty$ for some $t > 0$ is required. Our major goal in this note is the proof of the theorem:

Theorem 1. *If A is a Hermitian operator which is bounded below so that $D(A)$ contains a dense set of semi-analytic vectors, then A is essentially self-adjoint.*

This theorem is in the genre of the by-now classic theorem of Nelson [7] that a Hermitian operator with a dense set of analytic vectors is essentially self-adjoint (although, we shall see it lacks a certain deep aspect of Nelson's theorem). Theorem 1 is a special case of a theorem of Masson and McClary (Theorem 2, below) which was proven in a recent interesting paper [6]. To state their theorem, let us first recall two definitions:

Definition. A vector $\psi \in C^\infty(A)$ is called *quasi-analytic* if and only if

$$\sum_{n=1}^{\infty} \|A^n \psi\|^{-1/n} = \infty,$$

$\psi \in C^\infty(A)$ is called *Stieltjes* if and only if

$$\sum_{n=1}^{\infty} \|A^n \psi\|^{-1/2n} = \infty.$$

The relations of the four kinds of vectors is shown by the diagram: