

Linear Sections of Algebraic Varieties

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Introduction. The motivation for the topic studied in this paper is the observation that if V is a real non-singular algebraic variety in Euclidean space then a general family of parallel hyperplanes will contain a finite number of tangent hyperplanes. The section of V by one of these hyperplanes is not homeomorphic to neighboring sections in the parallel family, but if H is any hyperplane in this family other than the finitely many exceptions then the section $H \cap V$ is homeomorphic to neighboring sections, and in fact the homeomorphism is expressed more precisely by the fact that $H \cap V$ has a neighborhood in V of the form $(H \cap V) \times I$. The question raised here is whether this result would still hold if V were not non-singular. Intuitively one might expect it to be so, on the grounds that the singularities of neighboring sections should be similar to each other.

To see how one could attempt to prove this theorem on sections of any variety V , note first that the result is fairly trivial if the dimension of V is 1. So as a first nontrivial case consider a surface V with equation $f(x, y, z) = 0$ in 3-space, in relation to the family of plane sections with equations $x = \text{constant}$. The one-dimensional case can be applied to the branch curve D corresponding to the projection of V onto the (x, y) -plane. In fact something a bit stronger can be asserted. For if $x = 0$ is a line in the plane not tangent to D and not passing through any singular point of D , then it has a neighborhood in the plane, a strip parallel to the y -axis, fibered by arcs transversal to the lines $x = \text{constant}$, the arcs of D in the strip being included among the fibers. The idea now is to lift this fibering into the surface V , using its equation $f(x, y, z) = 0$.

To allow this lifting operation to work, it has to be known that each arc of the fibering in the plane will lift into a number of disjoint copies in V . And in order to see this it is not quite enough to know that the arc in the plane is either in D or not in D . A closer analysis of D is needed, taking into account the multiplicities of roots of f corresponding to points projection onto points of D . This leads to a study in the more general higher dimensional context of root distribution varieties (*cf.* Section 1).

Once this stage has been reached it is clear that the above idea gives a start

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