

# The Mean Ergodic Theorem and Saturation

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**1. Introduction.** The mean ergodic theorem on the behaviour of certain averages of the iterates  $T^n$  of  $T \in \mathcal{E}(\mathbf{X})$  (= the Banach algebra of endomorphisms of the Banach space  $\mathbf{X}$ ) in the strong operator topology of  $\mathcal{E}(\mathbf{X})$  for  $n \rightarrow \infty$  plays an important role in ergodic theory. This may be motivated by the fact that in general the limit  $\lim_{n \rightarrow \infty} T^n$  does not exist in any of the standard operator topologies of  $\mathcal{E}(\mathbf{X})$  (compare D. Ornstein [15]). As far as the authors are aware, all investigations connected with this theorem have been restricted to results concerning the convergence of the average in question; they have not dealt with the rate of this convergence. It is the purpose of this paper to equip some of the various types of the mean ergodic theorem with an order of approximation. The authors are grateful to Pesi Masani for sounding out some of the experts on ergodic theory during the Nice and Tihany meetings in September of 1970. Thanks are due to Richard M. Dudley for confirming this opinion and for pointing out the articles [5], [7] to the first-named author on the occasion of his Brandeis-Harvard-MIT Colloquium talk, 22 Oct. 1970, on the subject. Thanks are also due to Bruno Forte, Pavia, for valuable hints concerning the literature.

The ergodic theorem for the Cesàro average in the discrete case under the hypothesis of E. R. Lorch [14] (see N. Dunford-J. Schwartz [6, p. 662], for readily available proofs; see also the survey article on ergodic theory by P. R. Halmos [8]) states

**Theorem A (Lorch).** *Let  $T \in \mathcal{E}(\mathbf{X})$ ,  $\mathbf{X}$  being a reflexive Banach space, be a power-bounded operator, i.e.,  $\|T^n\| \leq M$ ,  $n = 0, 1, 2, \dots$  for some constant  $M$ . Then  $\{T^n; n \geq 0\}$  is strongly Cesàro-ergodic, i.e.*

$$(1.1) \quad (C, 1)\text{-}\lim_{n \rightarrow \infty} T^n f \equiv s\text{-}\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n T^k f = Pf$$

for each  $f \in \mathbf{X}$ , where  $P$  is the linear bounded projection (i.e.  $P^2 = P$ ) of  $\mathbf{X}$  onto the range  $\mathbf{R}(P) \equiv \mathbf{N}(I - T)$  parallel to the null space  $\mathbf{N}(P) \equiv \mathbf{R}(I - T)$ . Here,  $\mathbf{X} = \mathbf{N}(I - T) \oplus \mathbf{R}(I - T)$ .

The question now arises as to how rapidly the  $(C, 1)$ -means of the  $T^n f$  tend to  $Pf$  provided  $f$  belongs to some subspace of  $\mathbf{X}$ . It will turn out that the sub-