

Growth and Continuity of Functions Satisfying Quadratic Integral Inequalities

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1. Introduction. Let f be a real measurable function on $[0, 1]$. We are concerned with conditions on f of the form

$$(1.1) \quad I_p(f) = I(f) = \int_0^1 \int_0^1 \left[\frac{f(x) - f(y)}{p(x - y)} \right]^2 dx dy < \infty.$$

Here, p is a continuous function on $[-1, 1]$ such that

$$(1.2) \quad p(x) = p(-x) > 0 \text{ for } x > 0, \quad p \uparrow \text{ on } [0, 1], \quad p(0) = 0.$$

A. Garsia, E. Rodemich, and H. Rumsey [3, 4] considered a more general condition

$$(1.3) \quad I_{\Psi, p}(f) = \int_0^1 \int_0^1 \Psi \left(\frac{f(x) - f(y)}{p(x - y)} \right) dx dy < \infty,$$

where Ψ is a function on $(-\infty, \infty)$ satisfying

$$(1.4) \quad \Psi(x) = \Psi(-x), \quad \Psi \uparrow \text{ on } [0, \infty), \quad \lim_{x \rightarrow \infty} \Psi(x) = \infty.$$

Their theorem is

Theorem 1.1. *Assume that*

$$(1.5) \quad I_{\Psi, p}(f) \leq B,$$

$$(1.6) \quad \int_0^1 \Psi^{-1} \left(\frac{B}{t^2} \right) dp(t) < \infty.$$

Then we may make f continuous by modifying it on a set of measure 0. The modified f satisfies

$$(1.7) \quad |f(x) - f(y)| \leq 8 \int_0^{|x-y|} \Psi^{-1} \left(\frac{4B}{t^2} \right) dp(t)$$

for all x, y in $[0, 1]$.