

Interpolating Sequences for Bounded Harmonic Functions

JOHN GARNETT

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Introduction. A sequence $\{z_j\}_{j=1}^{\infty}$ in the open unit disc Δ is an *interpolating sequence* if for every bounded sequence $\{a_j\}_{j=1}^{\infty}$ there is a bounded analytic function f on Δ such that $f(z_j) = a_j$ for all j . If for every such $\{a_j\}_{j=1}^{\infty}$ there is a bounded harmonic function u on Δ with $u(z_j) = a_j$, we call $\{z_j\}_{j=1}^{\infty}$ a *harmonic interpolating sequence*. It is clear that every interpolating sequence is a harmonic interpolating sequence. Our purpose here is to prove the converse.

Theorem. *Every harmonic interpolating sequence is an interpolating sequence.*

The proof uses Hall's lemma and the construction from the proof of the corona theorem given in [2]. The theorem given here had also been proved by Carleson by similar means (unpublished).

§1. Two elementary lemmas. Fix a harmonic interpolating sequence $\{z_j\}_{j=1}^{\infty}$.

Lemma 1. *There are constants $M > 1$ and $\eta = \eta(M) > 0$ such that*

$$(1) \quad \left| \frac{z_j - z_k}{1 - \bar{z}_k z_j} \right| \geq \eta \quad \text{if } j \neq k,$$

(2) *If $|a_j| \leq 1$ for all j , there is u harmonic with $u(z_j) = a_j$ and $\|u\| = \sup |u(z)| \leq M$.*

Proof. The open mapping theorem for Banach spaces gives (2), and (1) is an easy consequence of (2). Indeed, let u be real and harmonic with $u(z_k) = 0$, $u(z_j) = 1$, $\|u\| \leq M$. If v is the conjugate harmonic function vanishing at z_k , then $f(z) = e^{u+iv} - 1$ satisfies $f(z_k) = 0$, $|f(z_j)| \geq e - 1$, and $\|f\| \leq e^M + 1$. By Schwarz's lemma,

$$\left| \frac{z_j - z_k}{1 - \bar{z}_k z_j} \right| \geq \frac{|f(z_j)|}{e^M + 1} \geq \frac{e - 1}{e^M + 1}.$$

For any sequence $\{z_j\}_{j=1}^{\infty}$ in Δ , let μ be the measure with mass $1 - |z_j|^2$ at z_j . Function theoretic properties of $\{z_j\}_{j=1}^{\infty}$ can often be expressed in terms of μ .