

The Sharp Maximum Principle for Nonlinear Inequalities

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Let $u(x)$ satisfy a second-order differential inequality with semidefinite quadratic form in a given region Ω . For a given point $P \in \Omega$, it is required to construct a set $S(P, \Omega)$ relative to which the maximum principle holds in the following sense: If $u(P) = \sup u(x)$ for $x \in S(P, \Omega)$, then necessarily $u(x) = u(P)$ for $x \in \tilde{S}(P, \Omega)$. Description of a suitable set $S(P, \Omega)$ and statement of the above fact about it constitute the *sharp maximum principle*. When $\tilde{S}(P, \Omega) = \Omega$ the sharp maximum principle reduces to the strong maximum principle of Bernstein-Hopf, and a different set $S(P, \Omega)$ gives the maximum principle of Nirenberg. The formulation above, which is vastly more general than either of these, was recently introduced by Hill. Hill's characterization of the set $S(P, \Omega)$ is retained here. The objective of this paper is to prove Hill's theorem (his proof is not quite complete) and at the same time extend it to broad classes of nonlinear inequalities. The analysis has many applications, some of which are indicated in the accompanying references. However, no applications are presented here.

1. The propagation set of Bony and Hill. Throughout this paper we assume

$$\alpha_{ij}(x) \in C^1, \quad \beta_i(x) \in C^1, \quad u(x) \in D^2, \quad v(x) \in D^2, \quad c(x) \in D^2$$

in a given region Ω of real Euclidean space E_n , though these assumptions could be weakened. All functions are real. A summation convention is used in which, unless otherwise stated, the indices i, j run from 1 to n while k and l run from 1 to m . Associated with α_{ij} and β_i are vector fields

$$X_k(x) = (\alpha_{ik}(x)), \quad X_0(x) = (\beta_i(x))$$

and differential operators

$$D_k = \alpha_{ik}(x) \frac{\partial}{\partial x_i}, \quad D_0 = \beta_i(x) \frac{\partial}{\partial x_i}.$$

According to Hill [2] the *diffusion trajectories* are curves C_k satisfying