

# *Local Isometric Embeddings of Surfaces into Euclidean Four Space*

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An old unsolved problem in differential geometry concerns the local isometric embedding of surfaces into three dimensional Euclidean space. Let  $g$  be a metric defined in a neighborhood of a point  $p \in \mathbf{R}^2$  and let  $E^3$  denote  $\mathbf{R}^3$  with the usual Euclidean metric. We ask if there exists a possibly smaller neighborhood  $U$  of  $p$  and a map  $f: U \rightarrow E^3$ ,  $f(x_1, x_2) = (f^1(x_1, x_2), f^2(x_1, x_2), f^3(x_1, x_2))$  such that the induced metric on  $U$  agrees with the original metric  $g$ . That is, we seek to solve the system of equations

$$\sum_{r=1}^3 \frac{\partial f^r}{\partial x_i} \frac{\partial f^r}{\partial x_j} = g_{ij}(x_1, x_2); \quad 1 \leq i \leq j \leq 2,$$

where  $g_{ij}$  are the components of the metric  $g$  with respect to local coordinates  $x_1, x_2$ . If such a map  $f$  exists, the metric  $g$  is called *realizable*. There are several reasons for excluding the case where the map  $f$  is only in  $C^1$ —any continuous metric can be realized by a  $C^1$  map [1] and very little differential geometry can be done with such embeddings.

It is not known if all metrics are realizable (*i.e.*, smoothly realizable). Affirmative results occur only in special cases; for instance, if the metric is analytic or the Gaussian curvature  $K$  of the metric is nonzero at  $p$ . Both of these results can be proved using a device introduced by Weingarten [2]. In this note we make the simple observation that this same device yields a proof that all two dimensional surfaces have local isometric embeddings into  $E^4$ . We do this by a reduction to the case of a metric with positive Gaussian curvature. We also provide a new proof that such metrics are realizable.

Pogorelov has recently constructed a  $C^{2,1}$  metric with no  $C^2$  realization in  $E^3$  (Soviet Math. Dokl. 12(1971), 729–731). This example may be modified to provide a  $C^{3+\alpha}$  metric with no such  $C^{2+\beta}$  realization,  $1 > \beta > 2\alpha > 0$ . This demonstrates the validity of our conjecture, contained in the Remark below, that additional smoothness occurs upon allowing an extra dimension.