

The Strong Szegő Limit Theorem for Circular Arcs

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Communicated by M. KAC

The strong Szegő limit theorem states that if f is a positive function on the unit circle whose derivative satisfies a Lipschitz condition (these conditions have been considerably relaxed in recent years) and $D_n[f]$ is the Toeplitz determinant

$$\det \left(\frac{1}{2\pi} \int_0^{2\pi} e^{i(j-k)\theta} f(\theta) d\theta \right), \quad j, k = 0, \dots, n-1,$$

then as $n \rightarrow \infty$ one has the asymptotic formula

$$D_n[f] \sim E[f]G[f]^n$$

where, if

$$\log f(\theta) \sim \sum_{k=-\infty}^{\infty} f_k e^{ik\theta},$$

one defines

$$G[f] = \exp f_0, \quad E[f] = \exp \frac{1}{4} \sum_{k=1}^{\infty} k f_k f_{-k}.$$

The analogue of this is obtained here for a function f which satisfies $f(\theta) = f(2\pi - \theta)$, which is supported on a closed arc $\alpha \leq \theta \leq 2\pi - \alpha$, and which when restricted to that arc is positive and has derivative satisfying a Lipschitz condition with exponent greater than $1/2$ (these conditions also can be relaxed). The formula reads

$$D_n[f] \sim 2^{1/12} e^{3\pi'(-1)} \left(\sin \frac{\alpha}{2} \right)^{-1/4} E[F]^2 n^{-1/4} G[F]^n \left(\cos \frac{\alpha}{2} \right)^n,$$

where E and G have the same meanings as before and

$$F(\theta) = f \left(2 \cos^{-1} \cos \frac{\alpha}{2} \cos \theta \right).$$