

The Obstruction to Fibering a Manifold over a Circle

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Introduction. Let M^n be a closed connected smooth manifold of dimension greater than five, and let $f : M \rightarrow S^1$ be a continuous map. The purpose of this paper is to give necessary and sufficient conditions for there to exist a smooth fiber map $\tilde{f} : M \rightarrow S^1$ homotopic to f .

One condition is that f_* is not the zero map, where $f_* : \Pi_1(M) \rightarrow \Pi_1(S^1)$. The necessity of this condition is seen by considering the homotopy exact sequence for a fiber space and remembering that the fiber should be a compact manifold. For convenience, we will assume that f_* is onto. This condition corresponds geometrically to the fiber of a smooth fibration \tilde{f} being connected. By the 1–1 correspondence between homotopy classes of continuous maps of $M \rightarrow S^1$ and elements of $H^1(M; Z)$, the above condition is equivalent algebraically to f being an indivisible element of $H^1(M; Z)$.

Let G denote the kernel of f_* and let X be the covering space of M corresponding to G . A second necessary condition is that X must be the homotopy type of a finite C. W. complex. This follows since X is the same homotopy type as the fiber of \tilde{f} , if \tilde{f} exists.

Let (N^{n-1}, ν) be a closed framed submanifold of M such that (N, ν) represents f under the Pontryagin–Thom construction, M_N denote the compact manifold obtained by “cutting” M along N . ∂M_N consists of two copies of N which we denote by N' and N'' . Hence, M_N is a cobordism from N' to N'' . The pair (N^{n-1}, ν) is called a splitting of M with respect to f . Let s be an integer smaller than $n - 2$ and larger than 1. Under the assumptions made thus far, we can always find a splitting (N, ν) such that (M_N, N) has a handlebody decomposition consisting of only s and $s + 1$ dimensional handles.

The existence of a smooth fiber map \tilde{f} is equivalent to the existence of a splitting (N, ν) such that M_N is diffeomorphic to $N \times [0, 1]$. There are two obstructions to the existence of such a splitting. The first obstruction, denoted by $c(f)$, is an element of an abelian group $C(Z(G), \alpha)$ and vanishes if and only if there exists a splitting (N, ν) such that (M_N, N) is an h -cobordism. (See Th. 5.2.) The second obstruction, denoted by $\tau(f)$, is defined if $c(f) = 0$ and is an