

A Converse of Taylor's Theorem

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Nelson [5] has given a simple proof of a converse of Taylor's theorem which had been proved earlier [1], [3], [4] by different means. It will be shown here that a modification of Nelson's approach yields a more general result and eliminates the need for a certain theorem of multilinear algebra.

We recall first some definitions and standard notation (see *e.g.* [2]). Let E, F be Banach spaces. Then $L(E, F)$ denotes the Banach space of continuous linear mappings from E into F with the usual operator norm, $L^k(E, F)$ denotes the Banach space of continuous k -multilinear mappings $v: E \times \cdots \times E \rightarrow F$ which may be identified with the spaces defined inductively as follows:

$$L^0(E, F) = F, L^k(E, F) = L(E, L^{k-1}(E, F)) = L^{k-1}(E, L(E, F)).$$

If $v \in L^k(E, F)$ and $(z_1, \cdots, z_k) \in E^k$, we shall sometimes write $v.z_1 \cdots z_k$ instead of $v(z_1, \cdots, z_k)$ and $v.z^k$ instead of $v(z, \cdots, z)$. The symmetrization Sv of v is defined by

$$Sv.z_1 \cdots z_k = \frac{1}{k!} \sum_{\sigma} v.z_{\sigma(1)} \cdots z_{\sigma(k)},$$

where the sum is taken over all permutations σ of the set $\{1, \cdots, k\}$.

Now let U be an open set in E . A mapping $f: U \rightarrow F$ is said to be differentiable at $x \in U$ if there exists a mapping $Df(x) \in L(E, F)$ such that

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - Df(x) \cdot h\|}{\|h\|} = 0.$$

In this case f is continuous at x and $Df(x)$, which is unique, is called the derivative of f at x . One defines in the usual manner higher order derivatives $D^k f: U \rightarrow L^k(E, F)$ and the notions of a C^k -mapping and a C^∞ -mapping.

The following statement is an immediate consequence of Taylor's theorem with an integral remainder (see [2]): if f is a C^r -mapping, $r \geq 0$, then for every $x_0 \in U$ there exist neighborhoods V of x_0 in U and W of 0 in E such that for every $(x, h) \in V \times W$

$$f(x+h) = \sum_{k=0}^r \frac{D^k f(x) \cdot h^k}{k!} + \rho(x, h),$$