

The Lusin-Menchoff Theorem in Metric Space

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Introduction. A number of authors have investigated the d -topology in Euclidean n -space E_n .

This topology is defined by Goffman and Waterman [1] as that topology in which the open sets are those Lebesgue measurable sets having each of its points as a point of density.

A point x is said to be a point of density of a measurable set A if

$$\lim_{d(I) \rightarrow 0} \frac{|A \cap I|}{|I|} = 1$$

where I is an oriented interval containing x , $d(I)$ is the diameter of I , and $|E|$ denotes the Lebesgue measure of E .

Goffman, Nishiura and Neugebauer [2] showed that if the I 's in the above definition are allowed to be arbitrary oriented intervals, then the d -topology is not completely regular. They showed further that if only those intervals having the ratio of the lengths of their sides bounded, above and below, are considered, then the d -topology is completely regular.

Using oriented squares, Troyer and Ziemer [3] proved complete regularity of the corresponding topology for outer Caratheodory measures defined on E_n .

Since then, the complete regularity of the d -topology has been used in various ways by Zink [4] and Ionescu-Tulcea [5, pp. 54-65]. It seems, therefore, that a generalization to more abstract spaces would be useful as well as intrinsically interesting.

In this paper, we prove the crucial lemma, the Lusin-Menchoff theorem, in metric space. A Morse halo condition [6] on a class of distinguished sets takes the place of regularity conditions on the intervals.

Our proof is simpler than Troyer and Ziemer's, yet it includes their result. Using the Lusin-Menchoff theorem and a lemma of Zahorski [7], the complete regularity of the d -topology in metric space follows as in the previous cases.

The Lusin-Menchoff theorem. Let Φ denote a finite regular outer Caratheodory measure on a metric space M with metric ρ . If $A \subset M$, then $d(A)$ will