

# *Two approximation properties and a Radon-Nikodym derivative for lattices of sets*

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**Abstract.** We show that the classical approximation property: an algebra  $\mathfrak{A}_1$  of sets is dense in the sigma algebra  $\mathfrak{A}_2$  which it generates when the topology is determined by a probability measure on  $\mathfrak{A}_2$ , is also valid for lattices of sets: a lattice  $\mathfrak{N}_1$  of sets is dense in the sigma lattice  $\mathfrak{N}_2$  which it generates. A second approximation property asserts that appropriate  $\mathfrak{N}_2$ -measurable functions can be approximated by simple  $\mathfrak{N}_1$ -measurable functions with respect to an Orlicz space norm satisfying the  $\Delta_2$ -condition. The two approximation properties and results of S. Johansen are then applied to obtain a Radon-Nikodym derivative for additive functions defined on lattices of sets.

**I. Introduction.** Throughout this paper,  $\mathfrak{N}$  will denote a lattice of subsets of a non-empty set  $\Omega$  (*i.e.*, both the empty set,  $\emptyset$ , and  $\Omega$  belong to  $\mathfrak{N}$ , and both  $A \cup B$  and  $A \cap B$  are in  $\mathfrak{N}$  whenever  $A$  and  $B$  are elements of  $\mathfrak{N}$ ).

A real valued function  $\mu$  on  $\mathfrak{N}$  is said to be additive if

- (i)  $\mu(\emptyset) = 0$  and
- (ii)  $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ ,  $A, B \in \mathfrak{N}$ .

Suppose each of  $\lambda$  and  $\mu$  is a bounded, additive function on  $\mathfrak{N}$  and  $\mu$  is non-negative (*i.e.*,  $\mu(A) \leq \mu(B)$  if  $A \subset B$ ).

We shall construct the Radon-Nikodym derivative of  $\lambda$  with respect to  $\mu$ . Denote by  $\mathfrak{A}$  the algebra of subsets of  $\Omega$  that  $\mathfrak{N}$  generates.

Results of B. J. Pettis [6, Theorem 1.2, p. 186] assert that each of  $\lambda$  and  $\mu$  has a unique additive extension to  $\mathfrak{A}$ . These extensions will also be denoted by  $\lambda$  and  $\mu$ . It is easy to check that  $\mathfrak{A}$  is the set of finite disjoint unions of elements of  $\mathfrak{F}$ , where

$$\mathfrak{F} = \{A \cap B^c; A, B \in \mathfrak{N}, B^c = \Omega - B\}.$$

Referring the reader to pages 97, 164–167, 240, 274–276, and 311–313 of Dunford and Schwartz [2] for details, we recall that there is a compact Hausdorff space  $\Omega_1$  and an isomorphism  $\tau$  of  $\mathfrak{A}$  onto the algebra  $\mathfrak{A}_1$  of closed and open