

# *Differential Operators of Infinite Order on a Lie Group, II*

ROE GOODMAN

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**Introduction.** Let  $\mathfrak{g}$  be a finite-dimensional complex Lie algebra, and  $\mathfrak{U}(\mathfrak{g})$  the universal enveloping algebra of  $\mathfrak{g}$ . In this paper, which is a continuation of [6], we construct certain complete topological algebras, each containing  $\mathfrak{U}(\mathfrak{g})$  as a dense subalgebra, and study the representations of these algebras obtained from Banach-space representations of  $G_0$  ( $G_0$  being a simply-connected real Lie group whose Lie algebra  $\mathfrak{g}_0$  is a real form of  $\mathfrak{g}$ ). Automorphisms of  $\mathfrak{g}$  extend continuously to automorphisms of these algebras, and we obtain results on the structure of these algebras as  $\text{Aut}(\mathfrak{g})$ -modules, generalizing the Poincaré–Birkhoff–Witt Theorem. We illustrate some applications of these algebras by constructing an operational calculus for elements of  $\mathfrak{U}(\mathfrak{g})$  and obtaining existence and uniqueness theorems for an abstract Cauchy problem.

In more detail, the contents of the paper are as follows: In §1 we construct a one-parameter family of Fréchet algebras  $\mathfrak{A}_\lambda(\mathfrak{g})$ ,  $0 < \lambda < \infty$ , each containing  $\mathfrak{U}(\mathfrak{g})$  as a dense subalgebra. The assignment  $\mathfrak{g} \rightarrow \mathfrak{A}_\lambda(\mathfrak{g})$  is functorial, and in §2 we show that when  $\lambda \geq 1$ , the space  $\mathfrak{A}_\lambda(\mathfrak{g})$  is isomorphic as an  $\text{Aut}(\mathfrak{g})$ -module to the space of entire functions on  $\mathfrak{g}'$  of exponential order  $1/\lambda$  and minimal type. In particular, the algebra  $\mathfrak{A}_1(\mathfrak{g})$  coincides with the “associative hyper-envelope”  $\mathfrak{F}(\mathfrak{g})$  constructed somewhat differently by P. K. Raševskii in [17] (cf. also [12]).

If  $\pi$  is a strongly continuous representation of the group  $G_0$  ( $G_0$  as above) on a Banach space  $\mathcal{H}(\pi)$ , and if  $S_\lambda(\pi)$  is the subspace of  $\mathcal{H}(\pi)$  defined in [6], then  $\pi$  defines a representation of the algebra  $\mathfrak{A}_\lambda(\mathfrak{g})$  on  $S_\lambda(\pi)$ , which is treated in §3. When  $\pi$  is the regular representation, either on  $L_p(G_0)$ ,  $1 \leq p < \infty$ , or on  $C_0(G_0)$ , and when  $\lambda > 1$ , then we obtain in this way a faithful representation of  $\mathfrak{A}_\lambda(\mathfrak{g})$  as “differential operators of infinite order” acting on  $S_\lambda(\pi)$ . In the case  $\lambda = 1$ , the algebra  $\mathfrak{A}_1(\mathfrak{g})$  can be faithfully represented acting on the space of germs of analytic functions at the identity of  $G_0$  [17]. (We do not know if the representation of  $\mathfrak{A}_1(\mathfrak{g})$  on the space of analytic vectors for the regular representation is faithful, in general.)