

On the Hyperboloidal Means and the Wave Equation

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Introduction. In a recent paper [1] hyperboloidal means HM were used to obtain representations of solutions of the wave equation $\square^2 w(X, t) = 0$ in the exterior of C_0 . In forming $HM[\zeta]$, the function ζ is integrated over $n - 1$ dimensional half-hyperboloids Hd which are the intersections of C_0 with time-like hyperplanes. In Sections 1 and 2 of this paper we introduce a class \mathfrak{R} of n dimensional surfaces (Hb) in the inside and on C_0 and construct on them measures which are invariant with respect to all proper Lorentz transformations. This gives an extension of the notion of HM , and we shall denote the new hyperboloidal means of functions Φ over Hb by $HS[\Phi]$.

Let $C_1(X, t)$ be the characteristic double cone with a vertex (X, t) which lies in the exterior of C_0 . Let (X, t) be fixed. Denote the upper cones of C_0 and C_1 respectively by C_0^+ and C_1^+ , and the corresponding closed solid cones by K_0 and K_1 . Set $\Omega = K_0 \cap K_1$. Inscribed in Ω there is a one parametric family $\mathfrak{F}(X, t) = \{Hb(\lambda), 0 < \lambda < \infty\}$ of half-hyperboloids $Hb(\lambda)$ of dimension n . Let $HS[\Phi; \lambda]$ be the mean of $\Phi(Y, \tau)$ over $Hb(\lambda)$. Suppose Φ is a function of class $C^{(2)}$ in K_0 with the origin deleted, and furthermore Φ vanishes properly at infinity. It is shown in Section 3 that

$$\frac{\partial}{\partial \lambda} HS[\Phi; \lambda] = \frac{1}{2\omega_n} (\frac{1}{4}(|X|^2 + \lambda^2 - t^2))^{-n/2} \int \square^2 \Phi d\tau dY$$

holds, where the volume integral extends over the interior of $Hb(\lambda)$, (Theorem 1).

If $\Phi = w$ is a solution of the wave equation in K_0 , then $HS[w; \lambda]$ is independent of λ , and one obtains a mean value theorem. Its applicability is restricted however to solutions which behave properly at infinity.

Let K'_0 denote the exterior of C_0 . In Section 4 of this paper we consider for each $(X, t) \in K'_0$ the n dimensional surface $Hb(X, t)$ consisting of that portion of C_0^+ which extends to infinity and is bounded below by $Hd(X, t) = C_0^+ \cap C_1^+(X, t)$. $Hb(X, t) \in \mathfrak{R}$. Let $\zeta(Y)$ be a smooth function on C_0^+ and denote by $HS[\zeta; X, t]$ the mean of ζ over $Hb(X, t)$. Then $w(X, t) = HS[\zeta; X, t]$ is a solution