

On the Total Variation of Solutions of Parabolic Equations

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Introduction. We shall be concerned with solutions of

$$(1) \quad u_t + a(t, x)u_x + \frac{1}{2}\sigma^2(t, x)u_{xx} = b(t, x)u + c(t, x)$$

defined in an open set U . The main result of this paper (Theorem 2) is an upper bound on the variation of u with respect to the variable x . The upper bound depends upon the boundary values, the functions b and c , and certain geometrical properties of the boundary itself, *i.e.*, it is an *a priori* estimate.

Explicit consideration of the variation of solutions is rare; other functionals such as the oscillation and various norms of the derivatives have been studied much more thoroughly. However, in some contexts the total variation arises naturally and bounds on the derivatives just won't do. A striking example is afforded by Oleinik's study [12] of a singular perturbation problem for a quasi-linear equation in which she proves convergence by a compactness argument. Since the limit function was discontinuous, a bound on the variation and not on the sup or on the L_2 norm of derivatives was needed. Another example would be the study of degenerate parabolic equations with bad coefficients.

Recently Sattinger [13] studied the variation of solutions of (1) when $b \equiv c \equiv 0$ and when σ^2 is bounded uniformly away from zero, *i.e.*, when the equation is uniformly parabolic. He considered separately the Cauchy Problem and the mixed initial-boundary value problem. In each case he showed that the variation was bounded by that of the data.

Even before Sattinger, however, Karlin and McGregor [7] had considered the equation $u_t + Au = 0$ where A is the generator of a one-dimensional diffusion process (cf. Ito and McKean, [5]). This equation includes as a special case the equation (1) when $b = c = 0$ and a and σ are independent of t . They proved that the fundamental solution for the pure Cauchy problem or mixed problem with constant boundary values is *totally positive*. But it is a basic property of totally positive kernels that the associated integral operator maps functions of bounded variation onto functions of bounded variation without