

# Measure Preserving Homeomorphisms of the Unit Disc

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We shall study homeomorphisms of the closed unit disc  $U$  onto itself which preserve the two dimensional Lebesgue measure  $m$  of any measurable set in  $U$ . Special emphasis is given to the investigation of the induced boundary correspondence.

The regularity conditions for the homeomorphisms  $h$  will successively be weakened. Starting off with diffeomorphism and bi-Lipschitzian mappings we are led to homeomorphisms with generalized partial derivatives and compositions thereof. All mappings considered will be sense preserving.

A homeomorphism  $h$  of the closed unit disc  $U$  onto itself is said to have boundary values  $\psi(\varphi)$  if  $h(e^{i\varphi}) = e^{i\psi(\varphi)}$ . If we say that the boundary homeomorphism  $\psi$  has certain properties, we mean that  $\psi$  considered as a function from  $R^1$  onto  $R^1$  has these properties. Clearly  $\psi(\varphi) - \varphi$  will always be periodic with period  $2\pi$ . Also, since  $\psi$  is a homeomorphism the derivative  $\psi' = d\psi/d\varphi$  exists a.e. and is a periodic function.

The simplest examples of measure preserving homeomorphisms are the rotations. These induce a boundary correspondence  $\psi(\varphi) = \varphi + \text{const.}$  They are the only mappings with Lipschitz constant 1 (a homeomorphism  $h$  with  $J_h = 1$  a.e. in  $U$  and satisfying  $|h(x) - h(x')| \leq |x - x'|$  for all  $x, x'$  in  $U$  must be conformal).

If the measure preserving homeomorphism  $h$  is Lipschitz-continuous with constant  $c \geq 1$  (i.e.  $|h(x) - h(x')| \leq c|x - x'|$  for all  $x, x'$  in  $U$ ), then the same holds for its inverse and also for its boundary correspondence. This follows easily from the fact that such a Lipschitz-continuous homeomorphism  $h$  is a.e. (totally) differentiable with Jacobian determinant equal to 1.  $h$  therefore is a  $c^2$ -quasiconformal mapping and the same must hold for its inverse  $h^{-1}$ . But then  $h^{-1}$  is a.e. differentiable and its Jacobian determinant is equal to 1 a.e. Hence  $h^{-1}$  is Lipschitz continuous with constant  $c$ .

Conversely if the boundary homeomorphism  $\psi$  and its inverse are Lipschitz-continuous with constant  $c$ , then there exists a measure preserving homeo-