

# *Estimates of Conformal Mappings Near the Boundary*

DIETER GAIER

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## §1. Introduction and statement of results.

**1.1. The Problem.** Let  $G$  be a finite, simply connected region with  $0 \in G$  and  $1 \in \partial G$ , and let  $f$  be the conformal map of  $G$  onto the disc  $D_w = \{|w| < 1\}$  with  $f(0) = 0$ ,  $f(1) = 1$ ; the meaning of  $f(1) = 1$  will become clear in a moment.

We are interested in the *problem*: If  $z$  is near  $z = 1$ , how near is  $w = f(z)$  to  $w = 1$ ?

This problem of the continuity of  $f$  near the boundary has been studied by various authors including Ferrand [3], Lavrientiev [7], and Warschawski [11], [12], [13]. However, the estimates of these authors do not contain numerically useful constants. It is the purpose of this paper to give estimates of the form  $|w - 1| < C |z - 1|^\mu$ , where  $C$  and  $\mu$  depend on geometric conditions. The order  $\mu$  is always sharp, and  $C$  is a small constant which in some cases is best possible.

Our proofs depend on estimates of harmonic measure which are interesting in themselves. We emphasize that we only study the map  $f$  from  $G$  onto  $D_w$ , not its inverse  $f^{-1}$ .

**1.2. Statement of the results.** Let  $G$  be as above. We say that  $z \in G$  is *visible* from a finite  $z_0 \in \partial G$ , if the open line segment  $\ell = (z, z_0)$  connecting  $z$  to  $z_0$  is contained in  $G$ , and we say that  $G$  is *starshaped* with respect to  $z_0 \in \partial G$ , if every  $z \in G$  is visible from  $z_0$ .

**Theorem 1.** *Assume that  $z \in G$  is visible from  $z = 1$  and that the function  $f$  mapping  $G$  onto  $D_w$  is normalized by  $f(0) = 0$  and  $f(t) \rightarrow 1$  as  $t \rightarrow 1$  on  $\ell = (z, 1)$ . Then, with  $w = f(z)$ , we have*

$$(1.1) \quad |w - 1| < 4|z - 1|^{1/2}.$$

*If, in addition,  $G \subset H$ , where  $H$  is a half plane with  $1 \in \partial H$ , we have*