

# *The Boundary Gradient Estimate for Quasilinear Elliptic and Parabolic Differential Equations*

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Let  $\Omega$  be a bounded domain in Euclidean  $n$ -space,  $n \geq 2$ . We shall write a second order, quasilinear differential operator, acting on functions  $u \in C^2(\bar{\Omega})$ , in the form

$$(1) \quad Qu = a^{ij}(x, u, u_k)u_{i,j} + a(x, u, u_k), \quad a^{ij} = a^{ji},$$

and assume that the coefficients  $a^{ij}$ ,  $1 \leq i, j, k \leq n$  lie in  $C^0(\bar{\Omega} \times E^{n+1})$ . We follow the customary convention that repeated indices indicate summation from 1 to  $n$  and have abbreviated derivatives by  $u_i = u_{x_i}$ ,  $u_{ij} = u_{x_i x_j}$  etc. The operator  $Q$  is *elliptic* if the coefficient matrix  $\|a^{ij}\|$  is positive in  $\Omega \times E^{n+1}$ .

It is well known by the fundamental work of Ladyzhenskaya and Ural'tseva [5] that the solvability of the Dirichlet problem for  $Q$  in  $\Omega$  depends upon the establishment of  $C^1(\bar{\Omega})$  estimates for the solutions of a related one parameter family of Dirichlet problems. The derivation of these estimates naturally splits into three stages: (i) estimates for  $\sup_{\Omega} |u|$ ; (ii) estimates for  $\sup_{\partial\Omega} |Du|$  in terms of  $\sup_{\Omega} |u|$ ; (iii) estimates for  $\sup_{\Omega} |Du|$  in terms of  $\sup_{\Omega} |u|$  and  $\sup_{\partial\Omega} |Du|$ . In certain cases, e.g. for operators of form  $Qu = a^{ij}(u_k)u_{i,j}$  the derivation of steps (i) and (iii) is trivial. The purpose of this work is to give an account of various sufficient conditions involving  $Q$  and  $\Omega$  under which the estimate (ii) may be carried out. The results are not substantially new and may largely be found in the union of the works [5], [7] and [9]. A fairly complete treatment of the elliptic situation, including demonstrations of the necessity of many of the conditions, is given by Serrin in [7]. Our aim here is primarily to present an alternative approach, retaining key ideas, which we believe provides some simplification of the important work of [7]. We also attempt some historical alignment by considering initially the situation that holds for convex or arbitrary  $C^2$  domains, roughly known before 1966, and from there covering the recent theory initiated by the work of Jenkins and Serrin on the minimal surface equation [4]. Finally we note some extensions to parabolic equations.