

Trace Norm Inequalities for the Measure of Hyponormal Spectra

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1. Let the bounded operator T on the Hilbert space \mathfrak{H} be hyponormal, so that

$$(1.1) \quad T^*T - TT^* = D \geq 0.$$

Then

$$(1.2) \quad \pi \|D\| \leq \text{meas}_2(\text{sp}(T));$$

Putnam [7], or, in case D is compact, Clancey [1]. Henceforth, it will be supposed that \mathfrak{H} is separable. This paper will deal with variations of (1.2) involving the (possibly infinite) trace of D . For properties of the trace of non-negative operators, see, *e.g.*, Plesner [5], vol. I, p. 225, Schatten [10], p. 77, [11], p. 36.

If T has the Cartesian form $T = H + iJ$, then (1.1) becomes

$$(1.3) \quad HJ - JH = -iC, \quad \text{where } D = 2C (\geq 0).$$

If H has finite maximum spectral multiplicity, then D is necessarily of trace class ($\text{tr}(D) < \infty$); see Kato [3]. It was recently shown in Putnam [9] that if H has a simple spectrum, then (1.2) can be sharpened to

$$(1.4) \quad \pi \text{tr}(D) \leq \text{meas}_2(\text{sp}(T)).$$

A generalization of (1.4) for arbitrary hyponormal operators T will be obtained below.

In the results to be obtained, there will be no loss of generality in supposing that T is completely hyponormal, so that there exists no non-trivial reducing space of T on which T is normal. It may be noted that a necessary condition for complete hyponormality of T is that any non-empty intersection of $\text{sp}(T)$ with an open set of the plane must have a positive Lebesgue planar measure; see [7]. In addition, the real (as well as the imaginary) part of T is absolutely continuous; [6], p. 42. Consequently, H is unitarily equivalent to, and will simply be taken to be equal to, the canonical representation described as follows.