

$$L(p, \infty)^*$$

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0. Introduction. $L(1, \infty)$, weak L^1 , was introduced in analysis when it was observed that some important operators (the Hilbert transform and the Hardy-Littlewood maximal function to name two) do not map L^1 into L^1 , but instead satisfy the weaker condition

$$\mu\{x: |(Tf)(x)| > y\} \leq C \frac{\|f\|}{y}.$$

It became natural therefore to investigate weak L^1 , the space of functions f satisfying

$$\mu\{x: |f(x)| > y\} < \frac{C}{y}.$$

This space was subsequently incorporated into a two-parameter family of spaces, $L(p, q)$, which also contains L^p spaces ($L(p, p)$ in this notation). The interpolation properties of $L(p, q)$ make them a very handy tool in some problems of harmonic analysis.

It is known that (except when taken over some trivial measure spaces) $L(1, \infty)$ is not normable. The question therefore arose as to whether any non-trivial continuous linear functionals on $L(1, \infty)$ exist. We show in this note that $L(1, \infty)^* \neq \{0\}$. It is worth noting that although for $1 < q < \infty$,

$$L(1, 1) \subset L(1, q) \subset L(1, \infty),$$

$L(1, q)^* = \{0\}$ when the measure space has no atoms (see [1]). Therefore in this case any c.l.f. on $L(1, \infty)$ necessarily vanishes on all simple functions.

For $1 < p < \infty$ it is known (see [1]) that $L(p, \infty)^* \supset L(p', 1)$. (Here and throughout this note $(1/p) + (1/p') = 1$.) We show that, except when the measure space consists of finitely many atoms, $L(p, \infty)^* \neq L(p', 1)$.

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