

Equivalences on Eigenvalues

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1. Introduction. Let m denote the set of complex $n \times n$ matrices. Let \mathfrak{H} , u , and \mathfrak{p} denote the set of self adjoint, unitary, and positive definite matrices in m . One of us [1] has recently proved

Theorem 1. *Let $X \in \mathfrak{p}$ and $Y \in \mathfrak{p}$. If $x_1 \geq \dots \geq x_n > 0$ and $y_1 \geq \dots \geq y_n > 0$ are the eigenvalues of X and Y , then*

$$(1) \quad (\exists U \in u) \text{ spec } Y = \text{spec } XU$$

\Leftrightarrow

$$\begin{aligned} x_1 &\geq y_1 \\ x_1 x_2 &\geq y_1 y_2 \\ &\vdots \end{aligned}$$

$$(2) \quad \begin{aligned} x_1 \cdots x_{n-1} &\geq y_1 \cdots y_{n-1} \\ x_1 x_2 \cdots x_n &= y_1 \cdots y_n, \end{aligned}$$

where $\text{spec } Y$ denotes the eigenvalues of Y , each counted with its proper multiplicity.

The main theorem of this paper is concerned with dropping the requirement that $Y \in \mathfrak{p}$.

Theorem 2. *Let $X \in \mathfrak{p}$ and $Y \in m$. If $x_1 \geq \dots \geq x_n > 0$ and y_1, \dots, y_n (normalized so that $|y_1| \geq \dots \geq |y_n| > 0$) are the eigenvalues of X and Y , then*

$$(3) \quad (\exists U \in u) \text{ spec } Y = \text{spec } XU$$

$$(4) \quad \Leftrightarrow$$

$$\begin{aligned} x_1 &\geq |y_1| \\ x_1 x_2 &\geq |y_1 y_2| \\ &\vdots \end{aligned}$$

$$\begin{aligned} x_1 \cdots x_{n-1} &\geq |y_1 \cdots y_{n-1}| \\ x_1 \cdots x_n &= |y_1 \cdots y_n|. \end{aligned}$$