

# A Note on Trudinger's Extension of Sobolev's Inequalities

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N. Trudinger [4] has shown that functions with bounded support in the unit ball of  $W_1^n(\mathbf{R}^n)$ , the space of  $L^n$  functions having first order derivatives in  $L^n$ , satisfy the inequality

$$\int (e^{\alpha|f(x)|^{n/n-1}} - 1) dx \leq c$$

for some positive constants  $\alpha, c$ . This is a substitute for boundedness, which holds for  $n = 1$  and fails for  $n > 1$ .

We establish a more general inequality for functions in  $L_{n/p}^p(\mathbf{R}^n)$ , the space of Bessel potentials of order  $n/p$  of  $L^p$  functions [2].

**Theorem.** *Functions with bounded support in the unit ball of  $L_{n/p}^p(\mathbf{R}^n)$ ,  $1 < p < \infty$ , satisfy the inequality*

$$\int (e^{\alpha|f(x)|^{p'}} - 1) dx \leq c$$

for some positive constants  $\alpha, c$  (depending on  $n, p$  and the support of  $f$ ), where  $p'$  is the dual index  $1/p + 1/p' = 1$ .

**Remarks.** Since  $L_1^n(\mathbf{R}^n)$  and  $W_1^n(\mathbf{R}^n)$  coincide and have equivalent norms, our result contains Trudinger's. We do not obtain Moser's refined estimate [1] for  $\alpha$ , however. Simple examples of the form  $|\log|x||^\alpha$  near  $x = 0$  show that the exponent  $p'$  cannot be improved. The same result holds for the spaces  $L_{n/p}^p(\Omega)$  for any bounded domain  $\Omega \subseteq \mathbf{R}^n$  satisfying a cone condition (see [3]).

*Proof of the Theorem.* We begin by proving

$$(*) \quad \|f\|_a \leq A \left(1 + \frac{q}{p}\right)^{1/q+1/p'} \quad \text{for } p \leq q < \infty.$$

Indeed by the definition of  $L_{n/p}^p$  we know  $f = G_{n/p} * g$  for  $g \in L^p$  with  $\|g\|_p \leq 1$ ,