

# *Invariant Para-closed Subspaces*

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In the study of lattices of invariant subspaces (suggested by P. R. Halmos), recent progress has been made (by D. E. Sarason, W. B. Arveson, H. Radjavi, P. Rosenthal, and others; see for instance [14], [1], [13], [12], [6]) in generalizing the classical Burnside theorem (see [18], Ch. I). Concerning these generalizations, we believe that an appropriate concept may be that of invariant para-closed subspaces (see Section 1 below), which seems related also to other topics in the theory of operators and which permits recovery of some of the above-mentioned results.

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## **Chapter I. Para-transitivity.**

1. Let  $\mathfrak{H}$  be a complex Hilbert space. A (linear) subspace  $\mathfrak{K} \subset \mathfrak{H}$  is, by definition, *para-closed* if it is the range  $A\mathfrak{R}$  of a linear bounded operator  $A$  from (perhaps) another Hilbert space  $\mathfrak{R}$ ; obviously, we may suppose  $\mathfrak{R} = \mathfrak{H}$  and  $A \geq 0$  (replacing  $A$  by  $(AA^*)^{1/2}$ ) or injective (replacing  $A$  by its restriction to  $\mathfrak{R} \ominus \ker A$ ). Obviously, any closed subspace is also para-closed. If  $\mathfrak{H}$  is of infinite dimension, then there exist subspaces which are not para-closed. (Take an orthonormal system  $\{\ell_n\}_{n=1}^\infty$  and the subspace generated by it.)

If  $\mathfrak{K}_1, \mathfrak{K}_2$  are para-closed subspaces of  $\mathfrak{H}$ , then so are  $\mathfrak{K}_1 \cap \mathfrak{K}_2$  and  $\mathfrak{K}_1 + \mathfrak{K}_2$ . Indeed, if  $\mathfrak{K}_j = A_j\mathfrak{R}_j$  ( $j = 1, 2$ ), then  $\mathfrak{K}_1 + \mathfrak{K}_2 = A\mathfrak{R}$  where  $A(k_1 \oplus k_2) = \{A_1k_1 + A_2k_2, k_1 \oplus k_2\}$  and  $\mathfrak{R} = \mathfrak{R}_1 \oplus \mathfrak{R}_2 = \mathfrak{R}$ , and  $\mathfrak{K}_1 \cap \mathfrak{K}_2 = A\mathfrak{R}_0$  where  $\mathfrak{R}_0 = \{k_1 \oplus k_2 \in \mathfrak{R}, A_1k_1 = A_2k_2\}$ .