

Existence of Multiple Solutions for Nonlinear Elliptic Boundary Value Problems

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1. Introduction. Let L denote a strongly uniformly elliptic linear second order differential operator and denote by B a first order boundary operator such that (L, B) satisfy a maximum principle. We consider nonlinear boundary value problems of the form

$$(1.1) \quad \begin{aligned} Lu &= f(x, u) \quad \text{in } \Omega, \\ Bu &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Boundary value problems of this type have attracted considerable interest during the last few years [2–14, 17, 18, 23, 25–27], in particular since they occur in chemical reactor theory and in nonlinear diffusion processes.

Under suitable assumptions it can be shown that (1.1) has at least one solution. Moreover, it can be shown that there exist a minimal solution u_1 and a maximal solution u_2 in a certain set S (e.g., in the set of nonnegative solutions) such that every solution in S satisfies $u_1 \leq u \leq u_2$.

However, even if u_1 and u_2 are distinct, nothing seems to be known about the existence of more than two solutions. In some specific problems the existence of more than two solutions has been conjectured by heuristic considerations [4, 7]. Furthermore, in case of the simpler problem where L is an ordinary second order differential operator the existence of multiple solutions has been proved by means of singular perturbation methods and isotonicity methods [8, 13, 17, 18]. However, these methods are essentially restricted to ordinary differential equations.

In this paper, for the first time, the existence of more than two solutions for the problem (1.1) will be shown. There will be given a non-constructive proof based on the theory of the Leray-Schauder degree [19]. The essential idea is the following: The problem (1.1) will be reduced to an equivalent Hammerstein equation in the space of continuous functions. Then, for large values of ξ the function $f(x, \xi)$ will be redefined in such a way that the corresponding nonlinear