

Semigroup Product Formulas, Compressions, and Continual Observations in Quantum Mechanics

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§1. Introduction. Let H be a Hilbert space with the inner product (which we take to be conjugate linear in the first variable) and norm denoted by $\langle \cdot \rangle_H$ and $\| \cdot \|_H$ or sometimes just $\langle \cdot \rangle$ and $\| \cdot \|$. We denote by $L(H)$ the bounded linear operators on H , and we call $A \in L(H)$ a contraction if its operator norm is ≤ 1 . Let $t \rightarrow F(t)$ be a strongly continuous mapping from the reals, \mathbf{R}^1 , (or the non-negative reals) to $L(H)$. We will consider the semigroup product formula:

$$(1) \quad \text{strong } \lim_{n \rightarrow \infty} (F(t/n))^n = \exp(tF'(0)).$$

Here $F'(0)$ is to be interpreted as some operator extension of strong $\lim_{t \rightarrow 0} t^{-1}(F(t) - I)$ (with domain all $u \in H$ on which the strong limit exists in H). For example, if A, B are self-adjoint operators in H , and $F(t) = \exp(itA) \exp(itB)$, then (1) is a formal statement of the Trotter Product Formula, $\text{strong } \lim_{n \rightarrow \infty} (\exp(itA/n) \exp(itB/n))^n = \exp(it(A+B))$ where $A+B$ must be properly interpreted as a self-adjoint operator in H . For another example, let E be an orthogonal projection in H , $F(t) = E \exp(itA)E$. Then (1) is formally $\text{strong } \lim_{n \rightarrow \infty} (E \exp(itA/n)E)^n = \exp(itEAE)$. Again EAE must be properly interpreted. If $H = L^2(\mathbf{R}^s)$,

$$A = H_0 = -\Delta = -\sum_{i=1}^s \frac{\partial^2}{\partial x_i^2}$$

and E is given by multiplication by the characteristic (indicator) function $\chi_{\mathcal{E}}$ of some closed subset \mathcal{E} of \mathbf{R}^s , then the last strong limit has an interesting physical interpretation. It is this example that we shall be most concerned with.

Formula (1) has been studied by Trotter ([8]) and Faris ([2]) in the case that $F(t) = \exp(itA) \exp(itB)$ and by Chernoff ([1]) in the general case, but to my