

On the Invertibility of a Class of Toeplitz Operators on the Quarter-Plane

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Let $L^2(\mathbf{T}^n)$ denote the usual Hilbert space of complex square integrable functions on the n -dimensional torus \mathbf{T}^n with respect to normalized Lebesgue measure. The Fourier transform of a function in $L^2(\mathbf{T}^n)$ is a complex function on the additive group \mathbf{Z}^n of ordered n -tuples of integers. The closed subspace $H^2(\mathbf{T}^n)$ of functions in $L^2(\mathbf{T}^n)$ whose Fourier transform is supported on the positive semigroup $\mathbf{Z}_+^n = \mathbf{Z}_+ \times \cdots \times \mathbf{Z}_+$ in \mathbf{Z}^n is of considerable importance in several areas of analysis. If P is the orthogonal projection of $L^2(\mathbf{T}^n)$ onto $H^2(\mathbf{T}^n)$ and φ is a continuous function on \mathbf{T}^n , then the Toeplitz operator W_φ on $H^2(\mathbf{T}^n)$ with symbol φ is defined by $W_\varphi f = P(\varphi f)$ for f in $H^2(\mathbf{T}^n)$.

The problem of understanding the nature of these operators and the operators in the closed algebra which they generate has stimulated much interest lately. In the case $n = 1$ the results are by now well-known (cf. [2]): the operator W_φ is a Fredholm operator if and only if φ doesn't vanish and has index equal to minus the winding number of the oriented curve determined by φ . Using results proved for convolution operators in smooth cones, Simonenko showed in [9] that W_φ is a Fredholm operator for φ on \mathbf{T}^2 if and only if φ doesn't vanish and is homotopic to a constant. (The assumption that the cone needs to be smooth was overlooked in [3], and hence the reasoning which implied that all Wiener-Hopf operators with non-vanishing symbol are invertible is invalid for $n > 2$.) Malyšev showed in [5] that for functions of the form

$$\varphi(z_1, z_2) = \sum_{j,k=-1}^1 \hat{\varphi}(j, k) z_1^j z_2^k,$$

the operator W_φ on $H^2(\mathbf{T}^2)$ is invertible if φ doesn't vanish and is homotopic to a constant and he gave a more or less explicit representation for the inverse (Malyšev considers the corresponding operator on the ℓ^1 space of functions in $H^2(\mathbf{T}^2)$ with an absolutely convergent Fourier series). A similar result was obtained by Osher in [7] for functions of the form $\alpha(z_1)\beta(z_2) + \gamma(z_1)$, where α , β , and γ are continuous.