

# *On Convexity and Compactness in F-Spaces with Bases*

JOEL H. SHAPIRO

*Communicated by* VICTOR KLEE

**Introduction.** This paper deals mainly with non-locally convex  $F$ -spaces, and contains variations on the following theme: *every closed, bounded, convex subset of  $\ell^p$  ( $0 < p < 1$ ) is compact*. More generally (Theorem 1), if  $E$  is an  $F$ -space with an absolute basis  $(e_n)$ , and the set  $\{te_n : t > 0\}$  is unbounded in the  $E$ -metric for each  $n$ , then the following three statements are equivalent:

(a) If  $(f_k)$  is a block basis for  $(e_n)$  which is bounded away from zero, then the series  $\sum t_n e_n$  diverges for some sequence  $(t_n)$  in  $\ell^1$ .

(b) No infinite dimensional subspace of  $E$  is locally convex.

(c) Every closed, metrically bounded, convex subset of  $E$  is compact.

A related result (Theorem 2) which depends only on the topology of  $E$ , and not on the particular metric that produces it, is the following: if  $E$  has a boundedly complete basis satisfying condition (a) above, then every closed, (topologically) bounded, convex subset of  $E$  is compact.

We apply these results to some sequence spaces related to the  $\ell^p$  spaces ( $0 < p < 1$ ). Specifically we consider the spaces  $\bigcap_{p > p_0} \ell^p$  ( $0 \leq p_0 < 1$ ); the spaces  $\ell^p$  of complex sequences  $f$  with

$$(0.1) \quad ||f|| = \sum \varphi(|f(n)|) < \infty,$$

where  $\varphi$  is a continuous, unbounded, subadditive, increasing function on  $[0, \infty)$ , with  $\varphi(t) = 0$  iff  $t = 0$ ; and the spaces  $\ell(p_n)$  of complex sequences  $f$  with

$$(0.2) \quad ||f|| = \sum |f(n)|^{p_n} < \infty,$$

where  $(p_n)$  is a sequence of numbers with  $0 < p_n \leq 1$ . Using Theorem 1 we show that whenever the function  $t^{-p}\varphi(t)$  is monotone decreasing on  $(0, \infty)$  for some  $0 < p < 1$ , then every closed, norm bounded, convex subset of  $\ell^p$  is compact; and that the same conclusion holds for  $\ell(p_n)$  iff  $\limsup p_n < 1$ . We show that if  $\lim p_n = 1$ , then every infinite dimensional subspace of  $\ell(p_n)$  has a further one isomorphic to a dense subspace of  $\ell^1$ . This is of interest because