

Cartan Matrices with Null Roots and Finite Cartan Matrices

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A Cartan matrix is an $n \times n$ matrix (A_{ij}) with integer coefficients which satisfies the conditions

- (i) $A_{ii} = 2, i = 1, 2, \dots, n,$
- (ii) $A_{ij} \leq 0$ if $i \neq j,$
- (iii) $A_{ij} = 0$ if and only if $A_{ji} = 0.$

We say that (A_{ij}) has a null root if there exists a non-zero column vector $[d_i] = [d_1, d_2, \dots, d_n]$ such that $(A_{ij})[d_i] = 0$, where each d_i is a non-negative integer. We call (A_{ij}) symmetrizable if there exists a non-singular diagonal matrix D such that the product $(A_{ij})D$ is a symmetric matrix.

Nowadays it is customary to represent Cartan matrices by diagrams which are a slight modification of the diagrams introduced by Coxeter to classify the discrete groups generated by reflections (see [3] or [5] chapter XI). Attached to a Cartan matrix there exists a group generated by reflections called its Weyl group. This is defined in Section 3 as well as the root system connected with it. When the matrix is symmetrizable the Weyl group leaves invariant a non-trivial quadratic form.

Cartan matrices are used in the construction of Lie algebras $\mathfrak{L}(A_{ij})$ over fields of characteristic 0; they generalize the matrices used by E. Cartan in his thesis of 1894 to classify the finite dimensional simple Lie algebras over the complex field. The result is that the algebra $\mathfrak{L}(A_{ij})$ is simple if (A_{ij}) is indecomposable in the sense defined in Section 1, and has no null roots; and it is finite dimensional if and only if it is one of the matrices obtained by Cartan. Because of this, we refer to these matrices as the finite Cartan matrices. The simplicity of $\mathfrak{L}(A_{ij})$ was established by Moody in [7] for symmetrizable matrices without null roots and in the general case by Kac in [6]. The extension of the construction to any field is carried out in [1], where it is found that with minor restrictions on the characteristic the result holds.