

# *Poised and Non-poised Hermite-Birkhoff Interpolation*

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Most familiar interpolation procedures, *e.g.*, Taylor, Lagrange and Hermite, are characterized by interpolation of function values and consecutive derivatives at one or more points. The more general problem of polynomial interpolation of scattered derivatives, including non-consecutive derivatives at several points, occur in such diverse contexts as the ascertainment of optimal quadrature formulae (see *e.g.*, Birkhoff [2], Schoenberg [12], Karlin [5]) and in establishing unicity of solutions to two-point boundary value problems for certain vibrating mechanical systems; in this connection, see Pólya's study of the beam equation [11].

G. D. Birkhoff [2] formulated such interpolation problems in 1906. Much of the current interest in the problem springs from I. J. Schoenberg's work [12], where he defined "Hermite-Birkhoff" interpolation involving the notion of an incidence matrix. Subsequently, contributions by several authors have delineated a wide class of problems which are *poised*, *i.e.*, for which Hermite-Birkhoff (H-B) polynomial interpolation is unique. We reviewed these findings in [7] and announced new characterizations of large classes of non-poised polynomial interpolation problems. Several results on H-B spline interpolation were also stated in that work.

In this paper, proofs of these results are provided. Section 1 contains the precise statement of H-B interpolation problems, for both polynomial and spline functions. Section 2 elaborates theorems on non-poised polynomial interpolation. The analysis includes a perturbation technique from which new collections of non-poised interpolation problems can be generated out of known non-poised interpolation problems (see Theorem 2.1). Important classes of non-poised interpolation problems are described in Theorems 2.2 and 2.3.

Section 3 presents criteria for unique polynomial interpolation when linear combinations of derivatives are prescribed. Section 4 is devoted to a simplified proof of Ferguson's theorem [3] on interpolation at complex points. After reviewing the concept of a spline function in Section 5, we develop two decomposition theorems (Section 6) relevant for spline interpolation. Some necessary conditions for unique interpolation (Section 7) and sufficient con-