

Conformal Metrics in \mathbb{R}^2 with Prescribed Curvature

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Let \mathfrak{M} be a Riemannian manifold with metric g . The following problem has recently been considered by a number of authors (see bibliography in [3]).

Problem I. What are necessary and sufficient conditions on a (smooth) function K for K to be the Gaussian curvature of a conformal metric $g' = e^{-2u}g$ on \mathfrak{M} ? This problem has been attacked by a number of authors using variational arguments ([1], [2], [3], [4], [7]). We would like to illustrate here how the problem can be attacked using the methods of [6]. We confine our remarks to problems on the plane, although the method may be extendable to the problem on general Riemannian manifolds.

The metric on the plane is $ds^2 = dx^2 + dy^2$ and we search for a metric of the form $ds^2 = e^{-2u}(dx^2 + dy^2)$ with prescribed curvature K on a given domain \mathfrak{D} in the plane. The curvature K is given by (see [5], p. 278) $K = e^{2u} \Delta u$. This may be written as

$$(1) \quad \Delta u - Ke^{-2u} = 0.$$

Thus, the problem of finding a Riemannian metric on a region \mathfrak{D} with prescribed curvature K is equivalent to finding a solution of (1) on \mathfrak{D} .

Theorem I. *If K is smooth and bounded and if \mathfrak{D} is bounded, then equation (1) has a solution, hence Problem I is solvable in this case.*

We shall solve this problem by using the technique of upper and lower solutions developed in [6]. Accordingly, it suffices to find two functions ϕ and ψ on \mathfrak{D} with the following properties:

- (i) $\psi < \phi$ on $\bar{\mathfrak{D}}$.
- (ii) $\Delta\psi - Ke^{-2\psi} \geq 0$ on $\bar{\mathfrak{D}}$.
- (iii) $\Delta\phi - Ke^{-2\phi} \leq 0$ on $\bar{\mathfrak{D}}$.