

Some Observations on Harmonic, Borel, Approximate, and L^p -Differentiability

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1. The notion of harmonic differentiability was introduced by E. M. Stein, see [2, p. 246], and proves to be an important tool in certain problems of differentiability. The definition of harmonic differentiability is the following. Let $f \in L^1(\mathbf{R}^n)$, $\text{supp } f \subset \{|x - x_0| \leq 1\}$, and let $u(x, y) = (P_y * f)(x)$, where $P_y(t) = c_n y / (|t|^2 + y^2)^{(n+1)/2}$ is the Poisson kernel in the upper half space $(t, y) \in \mathbf{R}_+^{n+1}$, i.e., $t \in \mathbf{R}^n$, $y > 0$. The constant

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}.$$

If u and $\partial u / \partial x_i$, $j = 1, \dots, n$, have non-tangential limits at x_0 , we say that f is harmonically derivable at x_0 .

The other notions of differentiability with which we are concerned are approximate, Borel, and L^p -differentiability. We say that a function f on \mathbf{R}^n is approximately derivable at x_0 iff there is $s: \mathbf{R}^n \rightarrow \mathbf{R}$ linear such that for each $\epsilon > 0$ the set $E_\epsilon = \{t: |f(x_0 + t) - f(x_0) - s(t)| \geq \epsilon |t|\}$ has 0 as a point of dispersion, and f is L^p -derivable at x_0 iff $\int_{|t| \leq h} |f(x_0 + t) - f(x_0) - s(t)|^p dt = o(h^{n+p})$. For $n = 1$, f has a Borel derivative L at x_0 iff $h^{-1} \int_0^h (f(x_0 + t) - f(x_0))/t dt \rightarrow L$ as $h \rightarrow 0$. It can be shown that L^p -differentiability at x_0 implies approximate and harmonic differentiability at x_0 . For the approximate case see [5] and for harmonic differentiability see [2, p. 247].

We will show in this paper that Borel differentiability implies harmonic differentiability, and from this we will deduce the perhaps surprising fact that harmonic differentiability does not imply approximate differentiability. Some further results on harmonic and L^p -differentiability are given, and the paper concludes with a generalization of the Fatou type theorem in [4].

2. Theorem 1. *If $f \in L^1(\mathbf{R}^1)$ has at x_0 a Borel derivative L , then f is harmonically derivable to L at x_0 .*