

# *Between Barreled and $W$ -Barreled for $C(X)$*

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1. For a completely regular Hausdorff space  $X$ , let  $C(X)$  be the set of continuous real-valued functions on  $X$ , with the compact-open topology. A basis for this topology is given by  $\{N(K, \epsilon, f): K \subset X \text{ is compact, } \epsilon > 0, f \in C(X)\}$ , where  $N(K, \epsilon, f) = \{g \in C(X): |f(t) - g(t)| \leq \epsilon \text{ for all } t \in K\}$ .

In 1954 L. Nachbin and T. Shirota proved independently that  $C(X)$  is barreled if and only if given any closed, non-compact subset  $F$  of  $X$  there exists  $f \in C(X)$  which is unbounded on  $F$  [2, 3].

A space with this property will be referred to as an  $N$ - $S$  space.

Wilansky has defined a linear topological space to be  $W$ -barreled if every closed, absorbing subset has non-empty interior [5, p. 224]. The purpose of this paper is to give a condition on  $X$  which is between  $C(X)$  barreled and  $C(X)$   $W$ -barreled. In Section 2 we define the term  $L$ - $W$  space, show that if  $X$  is an  $L$ - $W$  space,  $C(X)$  is barreled, show that if  $C(X)$  is  $W$ -barreled,  $X$  is an  $L$ - $W$  space, and give an example of a space  $X$  which is not an  $L$ - $W$  space for which  $C(X)$  is barreled. In Section 3 we study almost-discrete spaces in order to give an example of a space  $X$  which is an  $L$ - $W$  space but for which  $C(X)$  is not  $W$ -barreled. In Section 4 we introduce the concept of the center of a subset of  $C(X)$  and show its value for barrels and lack of value for  $W$ -barrels.

2. The main result in this section is Theorem 2.6 which gives a necessary condition on  $X$  for  $C(X)$  to be  $W$ -barreled.

**Definition 2.1.** A topological space  $X$  is said to be an  $L$ - $W$  space if for every decreasing sequence of closed, non-compact subsets  $\{F_n\}$  of  $X$  for which  $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$ , there exists  $f$  in  $C(X)$  unbounded on each  $F_n$ .

Clearly,  $C(X)$  is barreled whenever  $X$  is an  $L$ - $W$  space.

**Theorem 2.2.** *If  $C(X)$  is category II, then  $X$  satisfies the stronger condition that for every decreasing sequence  $\{F_n\}$  of closed, non-compact, non-empty subsets of  $X$  there exists  $f$  in  $C(X)$  unbounded on each  $F_n$ .*

*Proof.* For each  $n$ , let  $A_n = \{f \in C(X): |f(t)| \leq n \text{ for all } t \in F_n\}$ . Clearly