

A Note on a Nonlinear Elliptic Equation

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This note concerns the elliptic boundary value problem

$$(1) \quad \begin{cases} \mathcal{L}u \equiv \sum_{|\sigma| \leq 2m} a_\sigma(x) D^\sigma u = g(x, u), & x \in \Omega, \\ \mathcal{B}_i u \equiv \sum_{|\sigma| \leq m_i} b_{i\sigma}(x) D^\sigma u = 0, & x \in \partial\Omega, \end{cases}$$

where $1 \leq i \leq m$, $m_i < 2m$, $x = (x_1, \dots, x_n) \in \Omega$ which is a smooth bounded domain in \mathbf{R}^n , \mathcal{L} is uniformly elliptic [1] with smooth coefficients, g is continuous on $\bar{\Omega} \times \mathbf{R}$, the \mathcal{B}_i have smooth coefficients, and the usual multi-index notation is being employed. Moreover the boundary conditions are complementing in the sense of [1]. Under the above conditions on \mathcal{L} and the \mathcal{B}_i , \mathcal{L} is a Fredholm map on the space of functions satisfying the boundary conditions (in various topologies), i.e., \mathcal{L} has a finite dimensional kernel, a finite codimensional range in an appropriate space and the range is closed. In particular the Fredholm index of \mathcal{L} , $\chi = \dim \ker \mathcal{L} - \text{codim range } \mathcal{L}$ is finite.

We will obtain some existence results for (1) when $\chi > 0$ provided that g is odd in u . The oddness of g implies that $u = 0$ satisfies (1). In addition to this trivial solution (1) possesses infinitely many solutions when $\chi > 0$, in fact an unbounded component of solutions containing $u = 0$. Moreover a sharper estimate for the number of solutions of (1) can be given in terms of χ . This last statement follows from a general result (Theorem 14) from nonlinear functional analysis which is of independent interest. Precise existence statements for (1) are given in Theorems 9 and 11 below.

Our study of (1) was motivated by a recent paper of L. Nirenberg [2]. When certain technical conditions on \mathcal{L} , \mathcal{B}_i , and g were satisfied and $\chi \geq 0$, Nirenberg obtained a sufficient condition for the existence of solutions of (1). His criterion was in terms of the topological degree of a mapping when $\chi = 0$ and in terms of a generalized version of topological degree when $\chi > 0$. The $\chi = 0$ case contained an earlier result of Landesman and Lazer [3].

The various technical conditions of [2] are not needed here. Moreover our main tool is the usual Leray–Schauder degree theory together with the notion