

Strong and Weak Φ -Variation of Brownian Motion

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This paper is dedicated to Eberhard Hopf on the occasion of his seventieth birthday.

1. For a non-negative convex function Φ on $[0, \infty)$, with $\Phi(0) = 0$, a continuous real function on $[a, b]$ is said to be of bounded Φ -variation if

$$\sup \sum_{i=1}^{n-1} \Phi(|f(x_i) - f(x_{i-1})|) < \infty,$$

for all partitions $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$. This notion is of interest since there is a large set of functions Φ such that the Fourier series of every function of bounded Φ -variation converges uniformly.

For bounded variation itself, the variation may be obtained using any sequence of partitions whose norms converge to 0. This does not hold for the Φ -variation. If $\lim_{x \rightarrow 0} x^{-1}\Phi(x) = 0$, then for every continuous f there is a sequence of partitions, with norms converging to 0, for which the limit of the above sums is 0. This motivates a distinction between strong and weak Φ -variations and the following definitions. A function is of bounded strong Φ -variation if it satisfies the above condition (*i.e.*, if it is of bounded Φ -variation in the usual sense). It is of bounded weak Φ -variation if

$$\liminf \sum_{i=1}^{n-1} \Phi(\omega(f; [x_{i-1}, x_i])) < \infty,$$

where $\omega(f; [c, d])$ is the difference between the maximum and minimum of f on $[c, d]$ and the \liminf is taken for partitions with norms converging to 0.

Using $\Phi(x) = x^p$, $p \geq 1$, every continuous function f is given strong and weak variation dimensions $d_s(f)$ and $d_w(f)$. Then $d_w(f) \leq d_s(f)$, but we show that with probability 1, using Brownian motion, $d_w(f) = d_s(f) = 2$. The proof is obtained by use of a fine result of S. J. Taylor, [7], regarding the Hausdorff dimension of the graph of a continuous function, and an observation regarding the relation between Hausdorff dimension and weak variation dimension. Related remarks are made concerning Fourier series.