

The Lebesgue Set of a Function Whose Distribution Derivatives are p -th Power Summable

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This paper is dedicated to Eberhard Hopf on the occasion of his seventieth birthday.

1. Introduction. Assuming that $1 \leq p < n$, $s = np/(n - p)$ and f is a locally \mathcal{L}^n summable real valued function on \mathbf{R}^n whose distribution derivatives are p -th power locally summable, we prove here the existence of a set E with Hausdorff dimension at most $n - p$ such that to each point a in $\mathbf{R}^n \sim E$ corresponds a real number z for which

$$r^{-n} \int_{\{x: |x-a| < r\}} |f(x) - z|^s d\mathcal{L}^n x \rightarrow 0 \quad \text{as } r \downarrow 0.$$

Indeed we show that the functional capacity of degree p of the set E equals 0. For the special case $p = 1$ this result was announced in [5] and proved in [6, §4.5.9].

Various definitions and properties of functional capacity may be found in [1], [4], [7], [9], [10], [11], [12], [13]. However, we present here a simple and direct treatment of this topic, including several fundamental new facts.

Our method is based on geometric measure theory. We use the terminology of [6], which contains glossaries of basic notation on pages 669 to 671.

2. The space $\mathcal{Y}_p(\mathbf{R}^n)$ is the class of all $\bar{\mathbf{R}}$ valued \mathcal{L}^n measurable functions f such that

$$\int |f|^s d\mathcal{L}^n < \infty \quad \text{with} \quad \frac{1}{n} < \frac{1}{s} + \frac{1}{n} = \frac{1}{p} \leq 1$$

and there exist $\bar{\mathbf{R}}$ valued \mathcal{L}^n measurable functions u_1, \dots, u_n satisfying the conditions

$$\int |u_i|^p d\mathcal{L}^n < \infty \quad \text{and} \quad \int \varphi u_i d\mathcal{L}^n = - \int f D_i \varphi d\mathcal{L}^n$$

for all $i \in \{1, \dots, n\}$ and $\varphi \in \mathcal{D}^0(\mathbf{R}^n)$. It is a well known consequence that at