

Thinness and Wiener Criteria for Non-linear Potentials

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This paper is dedicated to Eberhard Hopf on the occasion of his seventieth birthday.

Introduction. In this work we will discuss the theory of thin sets relative to non-linear potentials of the form $u_\mu = k * (k * \mu)^{1/(p-1)}$. Here k is a kernel on \mathbf{R}^n , which, among other things, is radial and non-increasing in $|x|$, μ is a positive Radon measure and $1 < p < \infty$. Such potentials play a prominent role in the capacity theory of linear potentials $k * f$, where f is a function in the Lebesgue class \mathcal{L}_p ; see [7] and [8]. The continuity properties of the two kinds of potentials are closely tied, so that the theory of thin sets for the non-linear potential gives important information concerning the continuity of the linear potential. For a discussion of this see [6].

We distinguish two classes of non-linear potentials, $\mathcal{P}_{k,p}$ and $\mathcal{F}_{k,p}$. The first class contains more or less all potentials of interest, while the second is the subclass consisting of bounded potentials. We consider thinness relative to these two classes. If $E \subset \mathbf{R}^n$ and $x_0 \notin E$ we say that E is $\mathcal{P}_{k,p}$ -thin at x_0 , if $x_0 \notin \bar{E}$ or there exists $u = u_\mu \in \mathcal{P}_{k,p}$ such that

$$u(x_0) < \liminf_{\substack{x \rightarrow x_0 \\ x \in E}} u(x).$$

$\mathcal{F}_{k,p}$ -thinness is defined in the same way except that $u \in \mathcal{F}_{k,p}$.

Our principal concern is the Wiener Criterion for thinness. Here we impose the restriction that in the neighborhood of 0, $k(x)$ be comparable to a kernel $|x|^{\alpha-n}$, $0 < \alpha < n$. Further, we need consider only the case $\alpha p \leq n$, since, in the contrary case, the potentials are continuous functions. Pointwise upper and lower bounds for the non-linear potential, discussed at length in Section 3, play a crucial role. We will outline some of these results in rough form. Let $v(x_0, r) = \int_{\{|x-x_0|<r\}} d\mu(x)$. In all cases, $1 < p \leq n/\alpha$, we get the *lower* bound,

$$Q \int_0^{r_0} (r^{\alpha p - n} v(x_0, r))^{1/(p-1)} \frac{dr}{r} \leq u(x_0),$$