

# On Nonlinear Parabolic Variational Inequalities

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Let  $K$  be a closed convex subset of a real reflexive separable Banach space  $V$  with  $0 \in K$  and let  $A$  be a nonlinear monotone coercive operator mapping bounded sets of  $L^p(0, T; V)$  into bounded sets of  $L^q(0, T; V^*)$ . Brezis [2], Browder [4], Lions [8] and others have shown the existence of an element  $u$  in  $L^p(0, T; V)$  with  $u(t)$  in  $K$  for almost all  $t$ , such that

$$\int_0^T (v' + Au - f, v - u) dt \geq 0$$

for all  $v$  in  $L^p(0, T; V)$ ,  $v'$  in  $L^q(0, T; V^*)$ ,  $v(t)$  in  $K$  and  $v(0) = 0$ .

The main purpose of this paper is to study the regularity of the solution  $u$  when  $A$  is a nonlinear elliptic operator written in divergence form and  $V$  is a subspace of  $W^{m,p}(G)$ . Following Lions, we shall make a systematic use of the penalty function.

The notations and the assumptions are given in Section 1. The global regularity of  $u$  with respect to the time-variable is shown in Section 2. The results obtained, contain those of Brezis [2] and extend the local regularity results of Visik [11]. In Section 3, the case when  $K$  is a closed convex subset of both  $V$  and  $H$  with  $V \subset H$  is considered. With some additional hypotheses, the regularity of  $u$  with respect to the space-variables is obtained by using an imbedding theorem of Dubinskii [6].

The regularity of solutions of linear elliptic and parabolic inequalities has been studied by Lions [8], [9], Brezis–Stampacchia [3], Lewy–Stampacchia [7].

**Section 1.** Let  $G$  be a bounded open subset of  $E^n$  with a boundary  $\partial G$  of class  $C^\infty$ . Set  $D_i = i^{-1} \partial / \partial x_i$ ;  $j = 1, \dots, n$  and for each  $n$ -tuple  $\alpha = (\alpha_1, \dots, \alpha_n)$  of non-negative integers, we write

$$D^\alpha = \prod_{i=1}^n D_i^{\alpha_i} \quad \text{with} \quad |\alpha| = \sum_{i=1}^n \alpha_i.$$