

# Tangent Cones to Real Analytic Varieties

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**Introduction.** The usual notion of “tangent space” at a simple point of a real or complex analytic variety has been extended to “tangent cone” at an arbitrary point of the variety, simple or not (see, *e.g.*, [7, §§3–7], or [8, §§8–10]); in the complex case, Whitney in [7] and [8] has worked out a number of important properties of these tangent cones. We show in this paper that although these properties may not hold at certain points of a real analytic variety (Examples 1.6a–c), they must hold at each point off a real analytic subvariety of codimension  $\geq 2$  in the variety (Theorem 1.8).

## 1. Definitions; examples.

**Definition 1.1.** Let  $H$  be an open set in complex  $n$ -space,  $\mathbf{C}^n$ . A subset  $W$  of  $H$  is a *complex-analytic variety in  $H$*  if  $W$  is closed, and if at any point  $p \in W$ , there is a  $\mathbf{C}^n$ -open neighborhood  $U_p$  such that  $W \cap U_p$  is the set of common zeros of a collection of functions complex-analytic in  $U_p$ . A set  $V$  of real  $n$ -space  $\mathbf{R}^n$  is a *real  $\mathbf{C}$ -analytic variety in  $\mathbf{R}^n$*  if it is the real part of some variety  $W$  complex-analytic in some  $\mathbf{C}^n$ -open neighborhood of  $\mathbf{R}^n$ .

We shall refer to these simply as “*real-analytic varieties*”, or “*real varieties*”.

**Convention.** Throughout this paper, the letter “ $V$ ” will generally refer to the real case, the letter “ $W$ ” to the complex case; the letters “ $X$ ” and “ $Y$ ” will be used when consideration extends to both cases.

**Notation 1.2.** We denote the germ at  $p$  of a real or complex variety  $X$  by  $\tilde{X}_p$ .

For basic properties of real  $\mathbf{C}$ -analytic varieties and germs, see [11] or [4].

**Definition 1.3.** Let  $\tilde{X}_p$  be a real or complex analytic germ. The *geometric tangent cone to  $\tilde{X}_p$* , denoted  $T_v(\tilde{X}_p)$ , is the set of all vectors  $v$  satisfying the following condition:

There is a sequence of points  $\{p_i\}$ ,  $p_i \in X \setminus \{p\}$ ,  $p_i \rightarrow p$ , and a sequence of real numbers  $\{a_i\}$  such that  $v = \lim_{i \rightarrow \infty} a_i(p_i - p)$ .