

A PDE in \mathbf{R}^3 with Strange Behavior

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Analytic functions of one (or several) complex variables have certain well-known properties; *e.g.*, Liouville's theorem, unique continuation, the maximum modulus principle. It is a natural question to ask what becomes of these properties if, instead of analytic functions, one considers the class of functions annihilated by one (or several) smooth complex vector fields in \mathbf{R}^n . The purpose of this note is to discuss a very simple example of a smooth complex vector field in \mathbf{R}^3 ; it illustrates how the behavior of the coefficients in the equation can have a weird influence on the properties mentioned above.

The coordinates in \mathbf{R}^3 will be denoted by (t, y_1, y_2) . Let $a(t) \in C^\infty(\mathbf{R}^1)$ be a function $a : \mathbf{R}^1 \rightarrow \mathbf{R}^1$. The operator we consider is

$$L = \frac{\partial}{\partial t} + i \left(\frac{\partial}{\partial y_1} + a(t) \frac{\partial}{\partial y_2} \right), \quad i = (-1)^{1/2}.$$

By a solution of $Lu = 0$ in \mathbf{R}^3 , we shall mean a sufficiently smooth complex-valued solution, say $u \in C^\infty(\mathbf{R}^3)$.

We use the following terminology: L has the *Liouville property* on \mathbf{R}^3 iff every bounded solution of $Lu = 0$ in \mathbf{R}^3 is constant. L has the *unique continuation property* on \mathbf{R}^3 iff any solution of $Lu = 0$ in \mathbf{R}^3 , which vanishes on some nonvoid open set, must be identically zero. L satisfies the *maximum modulus principle* on \mathbf{R}^3 iff any solution of $Lu = 0$ in \mathbf{R}^3 , whose modulus has, at some point, a weak maximum relative to \mathbf{R}^3 , must be constant.

In what follows, let $\psi(t) = \int_0^t a(\tau) d\tau$ and $\Gamma = \text{graph } \psi$. The convex hull of a set A in \mathbf{R}^2 will be denoted by $\text{ch } A$.

Theorem. (a) L has the Liouville property on $\mathbf{R}^3 \Leftrightarrow \text{ch } \Gamma = \mathbf{R}^2$.

(b) L has the unique continuation property on $\mathbf{R}^3 \Leftrightarrow a(t)$ is not identically constant.

(c) L satisfies the maximum modulus principle on $\mathbf{R}^3 \Leftrightarrow \Gamma \subset \text{interior ch } \Gamma$.

Remarks. From (a) we see that the Liouville property depends only on the behavior of $a(t)$ at $t = \pm \infty$; *e.g.*, $a(t)$ can be arbitrary in any bounded interval,