

Hamiltonians for Singular Potentials

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1. Introduction. Quantum theory requires one to consider in $L^2(\mathbb{R}^3)$ operators of the form

$$(1.1) \quad L = -\Delta + V(x),$$

where Δ is the Laplacian and $V(x)$ is a function which is allowed to have singularities. We can define Lu for functions $u \in L^2$ such that Δu is in L^2 (in the sense of distributions) and Vu is also in L^2 . The theory requires that L thus defined have a "unique" (strictly) self-adjoint extension H , called the Hamiltonian. The best known result in this connection is that of Kato [13], who showed that L itself is (strictly) self-adjoint if V is real valued and in $L^2 + L^\infty$.

If L is not self-adjoint, the approach has been to find criteria for L to have a self-adjoint closure (we then call L essentially self-adjoint). Many authors have given criteria for this (see [7] for a partial list). The reason for the approach is that the closure is uniquely determined by L and consequently conforms with the requirements of the theory. However, one must require that the domain of L (i.e., the set of functions for which Lu is defined) is dense in L^2 . To accomplish this, it is usually assumed that V is locally in L^2 .

However, there are potentials V of interest in quantum mechanics which have such strong singularities that the domain of L is very small; indeed it may consist only of the function $u \equiv 0$. For such functions, L cannot have a self-adjoint closure. Hence there may be several self-adjoint extensions. The requirement that H depend uniquely on V causes a problem for such potentials.

Recently, Kato [7, p. 346], Schechter [4, 5], Faris [14] and Simon [12] have used a method of bilinear forms for defining the Hamiltonian (*cf.* Section 3). The basic idea is simple. Although Lu may not be in L^2 for many functions $u \in L^2$, the bilinear form

$$h(u, v) = (\nabla u, \nabla v) + (Vu, v)$$

is densely defined for V locally integrable. And under fairly mild conditions on V one can show that h determines a self-adjoint operator H which is an